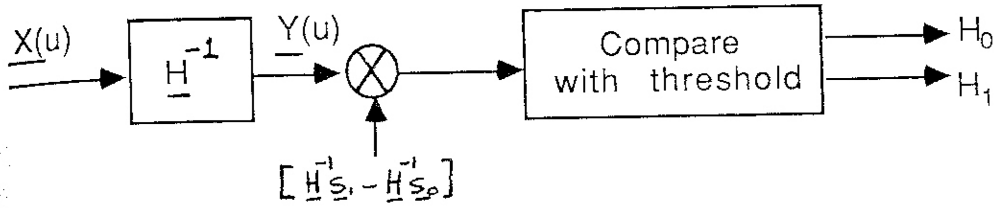

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$$\left|\underline{Y}(u) - \underline{H}^{-1}\underline{s}_1\right|^2 \underset{H_0}{\overset{H_1}{\lessgtr}} \left|\underline{Y}(u) - \underline{H}^{-1}\underline{s}_0\right|^2$$

$$\Leftrightarrow \quad |\underline{Y}(u)|^2 + |\underline{H}^{-1}\underline{s}_1|^2 - 2\underline{Y}(u)^T\underline{H}^{-1}\underline{s}_1 \underset{H_0}{\overset{H_1}{\lessgtr}} |\underline{Y}(u)|^2 + |\underline{H}^{-1}\underline{s}_0|^2 - 2\underline{Y}(u)^T\underline{H}^{-1}\underline{s}_0$$

$$\Leftrightarrow \quad \underline{Y}(u)^T[\underline{H}^{-1}\underline{s}_1 - \underline{H}^{-1}\underline{s}_0] \underset{H_0}{\overset{H_1}{\gtrless}} T = \frac{|\underline{H}^{-1}\underline{s}_1|^2 - |\underline{H}^{-1}\underline{s}_0|^2}{2}$$

In principle we can stop here. In addition this can be simplified by

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substitution as follows

$$\underline{x}^T\underline{H}^{-1^T}\underline{H}^{-1}(\underline{s}_1 - \underline{s}_0) \gtrless T$$

since $\underline{H}\underline{H}^T = \underline{K}_N$

$$\underline{x}^T\underline{K}_N{}^{-1}(\underline{s}_1 - \underline{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} T$$

where

$$\begin{aligned} T &= \frac{|\underline{H}^{-1}\underline{s}_1|^2 - |\underline{H}^{-1}\underline{s}_0|^2}{2} \\ &= \frac{\underline{s}_1^T\underline{K}_N{}^{-1}\underline{s}_1 - \underline{s}_0^T\underline{K}_N{}^{-1}\underline{s}_0}{2} \end{aligned}$$

i.e. the new version of the decision mechanism looks like this.

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Example :

$$H_1 \ : \ \underline{X}(u) = \begin{bmatrix} 0 \\ 10 \end{bmatrix} + N(u)$$

$$H_0 \ : \ \underline{X}(u) = \begin{bmatrix} 0 \\ -10 \end{bmatrix} + N(u)$$

where

$$\underline{m}_N = \underline{0} \ , \ \underline{K}_N = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

The above example has the following pictorial representation.
However $\underline{K}_N \neq \underline{I}$

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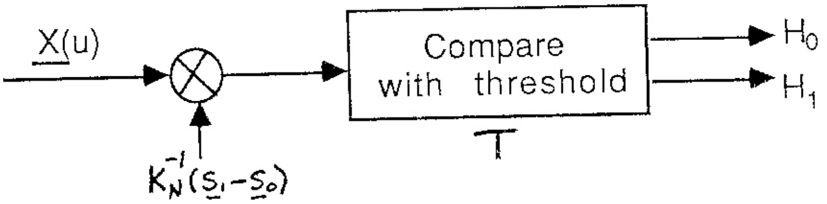
$$\Rightarrow \underline{K}_N{}^{-1} = \frac{1}{24} \begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}$$

and form

$$\underline{K}_N{}^{-1}(\underline{s}_1 - \underline{s}_0) = \frac{1}{24} \begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 20 \end{bmatrix} = \left(\frac{5}{3}\right) \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



$\underline{X}(u)$
Compare with threshold
H_0
H_1
T
$K_N^{-1}(\underline{s}_1-\underline{s}_0)$

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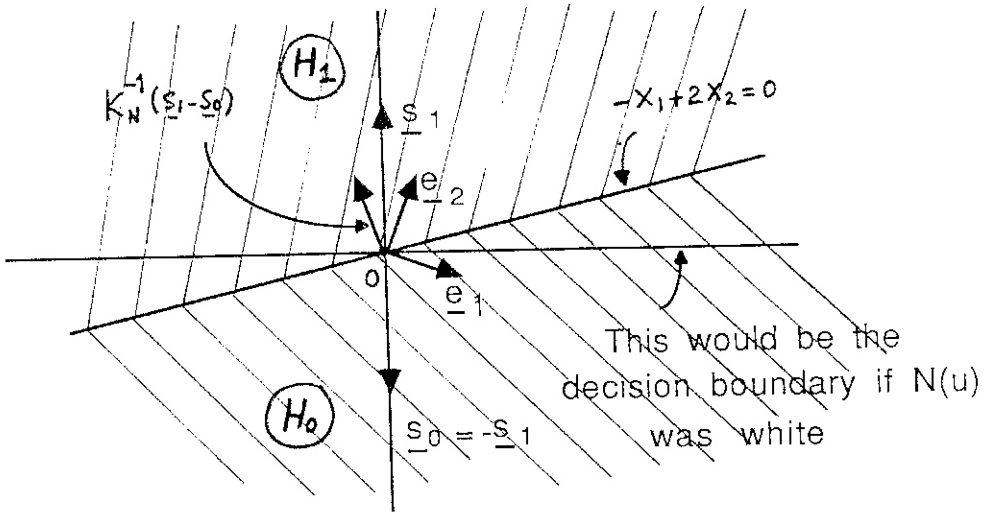

H_1
$K_N^{-1}(\underline{s}_1-\underline{s}_0)$
$\underline{s}_1$
$-x_1+2x_2=0$
$\underline{e}_2$
0
$\underline{e}_1$
This would be the decision boundary if N(u) was white
H_0
$\underline{s}_0=-\underline{s}_1$

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Also, $\underline{s}_1^T \underline{K}_N^{-1} \underline{s}_1 = \underline{s}_0^T \underline{K}_N^{-1} \underline{s}_0$

$$\Rightarrow T = 0$$

So the decision rule is

$$\underline{X}^T(u)\left(\frac{5}{3}\right)\begin{bmatrix} -1 \\ 2 \end{bmatrix} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

We can eliminate the constant factor to get the following decision rule

$$\Rightarrow -X_1(u) + 2X_2(u) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

To understand the *slope* of the decision boundary we write the K-L expansion of the noise :

$$\underline{K}_N = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

$$\lambda_1 = 3 \Rightarrow \underline{e}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 3 \Rightarrow \underline{e}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

i.e. $\underline{N}(u) = \sqrt{3}W_1(u)\underline{e}_1 + \sqrt{8}W_2(u)\underline{e}_2$. In order to perform the above, we have assumed that $\underline{K}_N^{-1}$ exists, that is the components of the noise are *not* linearly dependent. What would happen however if $\underline{K}_N$ were singular, i.e. if $\lambda_i = 0$?

This is the case of singular detection, in which a *perfect* decision can be made!

<u>Example</u> : Let us use the same signals as before

$$\begin{bmatrix} 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ -10 \end{bmatrix}$$

$$\underline{K}_N = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$$

$$\Rightarrow \underline{e}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix} \quad ; \quad \underline{e}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The two noise components are linearly dependent, i.e.

$$\underline{N}(u) = \sqrt{\lambda_2} W_2(u) \underline{e}_2 = W_2(u) \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} W_2(u) \\ 3W_2(u) \end{bmatrix}$$

So if we project along $\underline{e}_1$, which is the eigenvector with *zero* eigenvalue,

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we can make an unambiguous decision (i.e. with no errors) Since

$$\Rightarrow \underline{e}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

the decision rule is

$$(\underline{e}_1, \underline{X}) = -3X_1 + X_2 \gtrless 0$$

We have the following general statement:
Suppose $\underline{K}_N$ is singular, with $\lambda_1 = 0$ and some eigenvactor $\underline{e}_1$ corresponding to λ_1 . Then

$$\underline{N}(u) = \sum_{j=2}^{n} W_j(u) \sqrt{\lambda_j} \underline{e}_j$$

$$\underline{X}(u) = \underline{s}_i + \sum_{j=2}^{n} W_j(u) \sqrt{\lambda_j} \underline{e}_j$$

$$\Rightarrow \underline{X}^T(u) \underline{e}_1 = \underline{s}_i^T \underline{e}_1 + \sum_{j=2}^{n} \sqrt{\lambda_j} W_j(u) \underline{e}_j^T \underline{e}_1$$

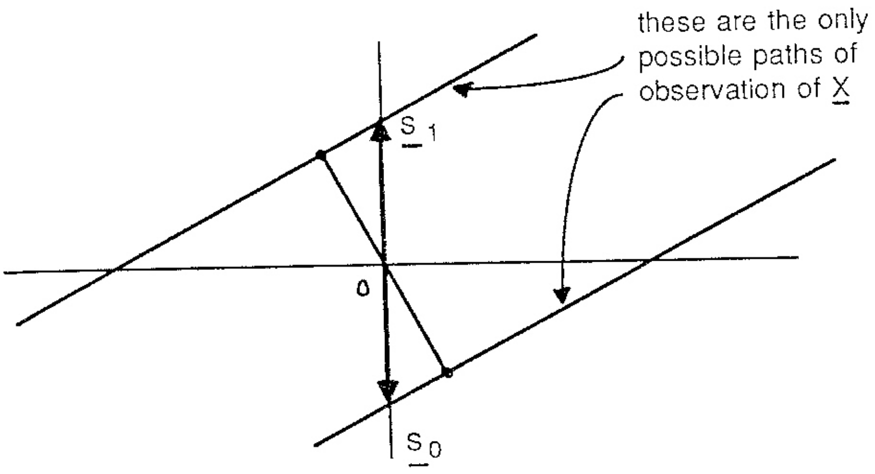
but $\underline{e}_j^T \underline{e}_1$ is zero by the orthogonality of the $\underline{e}_i$'s

$$\Rightarrow \underline{X}^T(u) \underline{e}_1 = \underline{s}_i^T \underline{e}_1$$

that means noiseless reception! Consequently if

$$\underline{s}_1^T \underline{e}_1 \neq \underline{s}_0^T \underline{e}_1$$

a perfect decision is possible.


these are the only
possible paths of
observation of $\underline{X}$
$\underline{S}_1$
0
$\underline{S}_0$

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