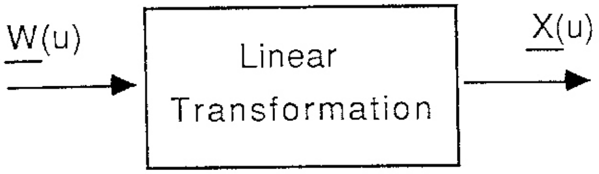

F2-1

Definition 2.2.2 *A complex (real) matrix* $\underline{A}$ *is called* unitary (orthogonal) *iff*

$$\underline{A}\underline{A}^{*T} = \underline{I}$$

Some facts about Hermitian symmetric matrices can be found in a textbook by R. Bellman, "Introduction to matrix analysis" Mc Graw Hill, 1970.

Theorem 2.2.1 *If* $\underline{K}$ *is Hermitian symmetric then there exists a unitary matrix* $\underline{E}$ *such that*

$$\underline{K} = \underline{E}\underline{\Lambda}\underline{E}^{*T}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

with λ_i *the eigenvalues of* $\underline{K}$, *not necessarily distinct.*

In other words, Hermitian symmetric matrices are always *diagonalizable*

Theorem 2.2.2 *A necessary and sufficient condition for such a* $\underline{K}$ *to be nonnegative definite is that* $\lambda_i \geq 0 \quad \forall i = 1, \ldots, n$

Theorem 2.2.3 *Let* $\underline{K}$ *be Hermitian symmetric. Then for each* <u>*distinct*</u> *(simple) eigenvalue there corresponds an eigenvector which is orthogonal to all others. To each eigenvalue of multiplicity* k *there correspond* k *linearly independent eigenvectors, which are orthogonal to all eigenvectors of other eigenvalues.*

However we can always perform a Gram-Schmit orthogonalization procedure and end up with k orthogonal eigenvectors. In summary, every Hermitian (n x n) matrix has n associated *orthogonal* eigenvalues $\{e_i\}_{i=1}{}^{n}$. In fact the matrix E of theorem 1 consists of these e_i's as its columns.

Returning to the factorization problem we want to find an $\underline{H} \ni \underline{R}_X = \underline{H}\underline{H}^{*T}$. We know that

$$\begin{aligned} \underline{R}_X &= \underline{E}\underline{\Lambda}\underline{E}^{*T} \\ &= \underline{E}\underline{\Lambda}^{1/2}\underline{\Lambda}^{1/2}\underline{E}^{*T} \end{aligned}$$

where

$$\underline{\Lambda}^{1/2} \stackrel{\text{def}}{=} \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sqrt{\lambda_n} \end{bmatrix} = (\underline{\Lambda}^{1/2})^{*T}$$

$$\begin{aligned}\Rightarrow \underline{R}_X &= \underline{E}\underline{\Lambda}^{1/2}\underline{\Lambda}^{1/2^{*T}}\underline{E}^{*T} \\ &= (\underline{E}\underline{\Lambda}^{1/2})(\underline{E}\underline{\Lambda}^{1/2})^{*T}\end{aligned}$$

i.e. we arrived a t solution where

$$\underline{H} = \underline{E}\underline{\Lambda}^{1/2}$$

There exists the question of whether this solution is unique or not. The answer is no. To see this take any unitary matrix $\underline{U} \ni \underline{U}\underline{U}^{*T} = \underline{I}$. Then

$$\begin{aligned}\underline{R}_X &= (\underline{E}\underline{\Lambda}^{1/2})\underline{I}(\underline{E}\underline{\Lambda}^{1/2})^{*T} \\ &= \underbrace{(\underline{E}\underline{\Lambda}^{1/2}\underline{U})}(\underline{E}\underline{\Lambda}^{1/2}\underline{U})^{*T} \\ &= another\underline{H}\end{aligned}$$

Sometimes we take $\underline{U} = \underline{E}^{*T}$ and the resulting

$$\underline{H} = \underline{E}\underline{\Lambda}^{*T}\underline{E}^{*T}$$

is called the " square root " of $\underline{R}_X$ since then $\underline{H}$ is Hermitian symmetric. From an applications viewpoint this is useful in simulation, i.e. creating a random vector with desired correlation properties, starting from a " random number generator ".

<u>**Note :**</u>

If $\underline{m}_X \neq \underline{0}$, then the appropriate linear transformation is

$$\underline{X} = \underline{H}\underline{W} + \underline{m}_X$$

where the factorization is done on $\underline{K}_X$, not $\underline{R}_X$.

<u>**Example :**</u>

Given the covariance matrix

$$\underline{K}_X = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}$$

we find the eigenvalues by solving

$$det(\underline{K}_X - \lambda_i\underline{I}) = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = \lambda_3 = 3/2$$

Solving for the eigenvectors we get

$$\lambda_1 = 0 \quad \Rightarrow \quad \underline{e}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3/2 \quad \Rightarrow \quad \underline{e}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\underline{e}_3 = \sqrt{\frac{2}{3}} \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}$$

So we could choose a linear transformation

$$\begin{aligned} \underline{H} = \underline{E}\underline{\Lambda}^{1/2} &= [\underline{e}_1 \mid \underline{e}_2 \mid \underline{e}_3] \begin{bmatrix} 0 & & 0 \\ & \sqrt{3/2} & \\ 0 & & \sqrt{3/2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \sqrt{3/2} & 1/2 \\ 0 & -\sqrt{3/2} & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

$$\Rightarrow \underline{X} = \underline{H}\underline{W} = \begin{bmatrix} 0 & \sqrt{3/2} & 1/2 \\ 0 & -\sqrt{3/2} & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} W(u,1) \\ W(u,2) \\ W(u,3) \end{bmatrix}$$

Notice that $\underline{X}$ does not depend on $W(u,1)$. In the above we solved the problem of spectral shaping which is equivalent to a covariance matrix factorization. The solution was unconstrained i.e. we imposed no restrictions on the nature of the linear transformation $\underline{H}$. We can pose an associated question : Can we find a *causal* matrix $\underline{H}$ for the same factorization job ?

Definition 2.2.3 *In this context, causal means lower triangular i.e.*

$$\begin{bmatrix} X(u,1) \\ \vdots \\ X(u,2) \end{bmatrix} = \begin{bmatrix} h_{11} & 0 & & 0 \\ h_{21} & h_{22} & & \\ \vdots & & \ddots & \\ h_{n1} & & \cdots & H_{nn} \end{bmatrix} \begin{bmatrix} W(u,1) \\ \vdots \\ W(u,n) \end{bmatrix} + \underline{m}_X$$

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or

$$X(u,i) = \sum_{j=1}^{i} h_{ij} W(u,j)$$

This is called the *Cholesky Decomposition of positive definite matrices. We restate the problem as follows: Find a lower-triangular matrix $\underline{H}$ such that*

$$\underline{K}_X = \underline{H}\underline{H}^{*T}$$

<u>**Example :**</u>

For the real case

$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{12} & \cdot & & \\ \vdots & & \ddots & \\ k_{1n} & & \cdots & k_{nn} \end{bmatrix} = \begin{bmatrix} h_{11} & 0 & \cdots & 0 \\ h_{21} & h_{22} & & \\ \vdots & & \ddots & \\ h_{n1} & & \cdots & h_{nn} \end{bmatrix} \begin{bmatrix} h_{11} & h_{21} & \cdots & h_{n1} \\ 0 & h_{22} & & \\ 0 & & \ddots & \vdots \\ 0 & & & h_{nn} \end{bmatrix}$$

From ${h_{11}}^2 = k_{11} \quad \Rightarrow \quad h_{11} = \pm\sqrt{k_{11}}$

$$\Rightarrow k_{12} = h_{21}h_{11} \quad \Rightarrow h_{21} = \frac{k_{12}}{h_{11}}$$

In the same manner we can find the rest of the h_{ij}.

Using the concept of covariance factorization we can derive a set of insightful properties:

i. Spectral resolution

   Assume a real covariance matrix $\underline{K}_X$. From the theory we know that $\underline{K}_X$ can be decomposed as

$$\underline{K}_X = \underline{E}\underline{\Lambda}\underline{E}^T$$

   where

$$\underline{E} = [\underline{e}_1 \mid \underline{e}_2 \mid \cdots \mid \underline{e}_n]$$

   is the matrix of *orthonormal* eigenvectors of $\underline{K}_X$ and

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

is the diagonal matrix of nonnegative eigenvalues. We can rewrite this as

$$\underline{K}_X = [\lambda_1 \underline{e}_1 \mid \lambda_2 \underline{e}_2 \mid \cdots \mid \lambda_n \underline{e}_n] \begin{bmatrix} \underline{e}_1{}^T \\ -- \\ \underline{e}_2{}^T \\ \vdots \\ -- \\ \underline{e}_n{}^T \end{bmatrix}$$

or

$$\underline{K}_X = \sum_{i=1}^{n} \lambda_i \underline{e}_i \underline{e}_i{}^T$$

This shows that $\underline{K}_X$ can be decomposed (resolved) into a sum of n matrices, each of the form $\underline{e}_i \underline{e}_i{}^T$, with weight λ_i. The set of n eigenvectors $\{\underline{e}_i\}_{i=1}^n$ constitutes a basis for the n-dimensional vector space and each deterministic vector $\underline{A}$ can be expanded into a series

$$\underline{A} = \sum_{i=1}^{n} a_i \underline{e}_i$$

where

$$a_i = (\underline{A}, \underline{e}_i) = \underline{A}^T \underline{e}_i$$

So far we have determined that given *some covariance matrix* $\underline{K}_X$, *we can find its n orthonormal eigenvectors* $\underline{e}_i$ *and use them as a basis of an n-dimensional vector space. Each* deterministic vector $\underline{A}$ can be described in terms of its " projections " a_j along the $\underline{e}_j$ coordinate. It is also clear that we can create random vectors by choosing these projections to be random variables $A_i(u)$, i.e.

$$\underline{A} = \sum_{i=1}^{n} A_i(u) \underline{e}_i$$

<u>Note :</u> If the eigenvectors have the form $\underline{e}_i = [0 \ \cdots \ 1 \ \cdots \ 0]^T$ with 1 in the i^{th} position then

$$\underline{A} = \begin{bmatrix} A_1(u) \\ \vdots \\ A_n(u) \end{bmatrix}$$

The question arises whether these random coefficients $A_i(u)$ can be chosen in such a way that the resulting $\underline{A}$ is actually $\underline{X}(u)$ i.e. the vector whose covariance is the given one $\underline{K}_X$. We will answer that later (whitening). Note that whitening is the reverse problem to " coloring " or " spectral shaping ".