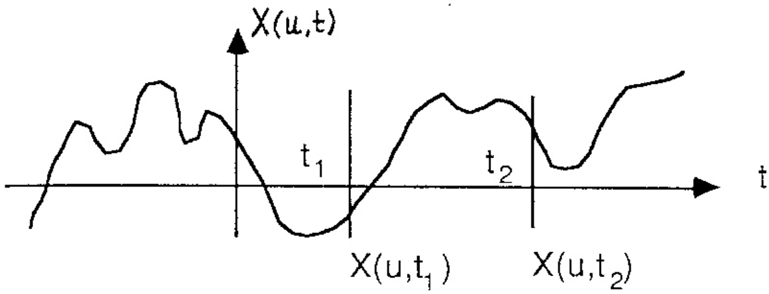

F1-5

2. Random sinusoid

$$X(u,t) = sin(t + \phi(u))$$

$$\begin{aligned}
\mathcal{E}\{X(u,t)\} &= \int_0^{2\pi} sin(t+\phi) f_\phi(\phi) d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} sin(t+\phi)d\phi = 0 \qquad \forall t
\end{aligned}$$

$$\begin{aligned}
R_X(t_1,t_2) &= \mathcal{E}\{sin(t_1+\phi)sin(t_2+\phi)\} \\
&= \frac{1}{2\pi}\int_0^{2\pi} sin(t_1+\phi)sin(t_2+\phi)d\phi = 0 \\
&= \frac{1}{2\pi}\frac{1}{2\pi}\int_0^{2\pi} [cos(t_1-t_2) - cos(t_1+t_2+2\phi)]d\phi \\
&= \frac{1}{2}cos(t_1-t_2)
\end{aligned}$$

Notice that the autocorrelation depends only on the *difference* $t_1 - t_2$ and n t_1 or t_2 individually.
It is natural to ask the question : Are there any restrictions (or propertie imposed on a function $R_X(t_1,t_2)$ in order for it to be a *legitimate* correlati function ?

Example 1

Can $R_X(t_1,t_2) = -1$ be a correlation function ?
It holds that $X^2(u,t_1) \geq 0$. So $R_X(t_1,t_1) = \mathcal{E}\{X^2(u,t_1)\} \geq 0$ Then -1 is r a legitimate choice.

Example 2

What about the function with $K_X(t_1,t_1) = 0.8$, $K_X(t_1,t_2) = -1, t_1 \neq t_2$?

We list below some properties than any autocorrelation or covariance funct must satisfy.

(a) Any well defined function $m_X(t)$ can be the mean function $\mathcal{E}\{X(u,t)\}$ of a process

(b) the correlation function $R_X(t_1,t_2)$ must be *hermitian symmetric* i.e.

$$R_X(t_1,t_2) = R_x{}^*(t_2,t_1)$$

Proof :

by definition

$$\begin{aligned} R_X(t_1,t_2) &\stackrel{\text{def}}{=} \mathcal{E}\{X(u,t_1)X^*(u,t_2)\} \\ &= \mathcal{E}\{X^*(u,t_2)X(u,t_1)\} \\ &= \mathcal{E}\{[X(u,t_2)X^*(u,t_1)]^*\} \\ &= \mathcal{E}^*\{X(u,t_2)X^*(u,t_1)\} \qquad qed \end{aligned}$$

(c) The correlation function must be a *nonnegative definite* function :

Definition 1.3.1 *A complex function $R_X(t_1,t_2)$ is called nonnegative definite iff for any choice of n complex numbers $a_1, a_2, \ldots, a_n$ and every n-tuple $(t_1, t_2, \ldots, t_n)$, it is true that*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a^*{}_j R_X(t_i,t_j) \geq 0 \qquad \otimes$$

Proof :

We prove the necessary part, i.e. that if $R_X(t_1,t_2)$ *is* a correlation function for some process $X(u,t)$, then $\otimes$ holds

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a^*{}_j R_X(t_i,t_j) =$$

$$\begin{aligned} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a^*{}_j \mathcal{E}\{X(u,t_i)X^*(u,t_j)\} \\ &= \mathcal{E}\{|\sum_{i=1}^{n} a_i X(u,t_i)|^2\} \geq 0 \end{aligned}$$

The converse will be proven later.

(d) The correlation function must satisfy the Schwartz inequality

$$|R_X(t_1,t_2)| \leq \sqrt{R_X(t_1,t_1)}\sqrt{R_X(t_2,t_2)}$$

Proof :

For the real case we must show that

$$R_X{}^2(t_1,t_2) \leq R_X(t_1,t_1)R_X(t_2,t_2)$$

Consider

$$\mathcal{E}\{[X(u,t_1)+aX(u,t_2)]^2\} \geq 0$$

for any a $\in \mathcal{R}$

$$\Rightarrow \mathcal{E}\{X^2(u,t_1)+a^2X^2(u,t_2)+2aX(u,t_1)X(u,t_2)\} \geq 0$$

$$\Rightarrow R_X(t_1,t_1)+a^2R_X(t_2,t_2)+2aR_X(t_1,t_2) \geq 0$$

The above holds for all a! Viewed as a parabola with respect to a, the above binomial is negative if its discriminant is nonpositive i.e.

$$4R_X{}^2(t_1,t_2) - 4R_X(t_1,t_1)R_X(t_2,t_2) \leq 0$$

Note that in example 2 above, teh function is illegitimate because

$$|-1| \not\leq \sqrt{0.8}\sqrt{0.8}$$

We can express $R_{ZZ^*}(t_1,t_2)$ of a complex process

$$Z(u,t) = X(u,t) + jY(u,t)$$

in terms of real quantities namely the autocorrelation and cross correlation of its real and imaginary parts as follows

$$\begin{aligned}
R_{ZZ^*}(t_1,t_2) &= \mathcal{E}\{Z(u,t_1)Z^*(u,t_2)\} \\
&= \mathcal{E}\{(X(u,t_1)+jY(u,t_1))(X(u,t_2)+jY(u,t_2))^*\} \\
&= [R_{XX}(t_1,t_2)+R_{YY}(t_1,t_2)] + j[R_{YX}(t_1,t_2) - R_{XY}(t_1,t_2
\end{aligned}$$

Notice that $R_{YX}(t_1,t_2) = R_{XY}(t_1,t_2) \neq R_{XY}(t_1,t_2)$ i.e. the symmetr property does not hold for *crosscorrelations*. One could ask the followin question : Why not define

$$R_{ZZ}(t_1,t_2) = \mathcal{E}\{Z(u,t_1)Z(u,t_2)\}$$

and call *that* a correlation function for a complex process $Z(u,t)$? Th reason is that this is *not* a nonnegative definite function. In the nex chapter we extend these notions to random vectors.

The correlation function must satisfy the Schwartz inequality

$$|R_X(t_1,t_2)| \leq \sqrt{R_X(t_1,t_1)}\sqrt{R_X(t_2,t_2)}$$

Proof :

For the real case we must show that

$$R_X{}^2(t_1,t_2) \leq R_X(t_1,t_1)R_X(t_2,t_2)$$

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in terms of real quantities namely the autocorrelation and cross correlation of its real and imaginary parts as follows

$$\begin{aligned}
R_{ZZ^*}(t_1,t_2) &= \mathcal{E}\left\{Z(u,t_1)Z^*(u,t_2)\right\} \\
&= \mathcal{E}\left\{(X(u,t_1)+jY(u,t_1))\left(X(u,t_2)+jY(u,t_2)\right)^*\right\} \\
&= \left[R_{XX}(t_1,t_2)+R_{YY}(t_1,t_2)\right] + j\left[R_{YX}(t_1,t_2)-R_{XY}(t_1,t_2)\right]
\end{aligned}$$

Notice that $R_{YX}(t_1,t_2) = R_{XY}(t_1,t_2) \neq R_{XY}(t_1,t_2)$ i.e. the symmetry property does not hold for *crosscorrelations*. One could ask the following question : Why not define

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and call *that* a correlation function for a complex process $Z(u,t)$? The reason is that this is *not* a nonnegative definite function. In the next chapter we extend these notions to random vectors.

Chapter 2

RANDOM VECTORS

2.1 Definition - Correlation and Covariance matrix

Random vectors come about either by "sampling" one random process $X(u,t)$ at $t_1, t_2, \ldots, t_n$ or by "observing" a number of processes $X_1(u,t), X_2(u,t), \ldots, X_n(u,t)$ at the same time. Essentially the two ways are equivalent mathematically. Let

$$\underline{X}(u) \stackrel{\text{def}}{=} \begin{bmatrix} X(u,1) \\ \vdots \\ X(u,n) \end{bmatrix} \qquad \mathcal{E}\{\underline{X}(u)\} \stackrel{\text{def}}{=} \begin{bmatrix} m_X(1) \\ \vdots \\ m_X(n) \end{bmatrix}$$

Then the autocorrelation function is

$$\begin{aligned} \underline{R}_X &= \mathcal{E}\{\underline{X}(u)\underline{X}^{*T}(u)\} \\ &= \mathcal{E}\left\{\begin{bmatrix} X(u,1) \\ \vdots \\ X(u,n) \end{bmatrix} [X(u,1), \ldots, X(u,n)]^*\right\} \\ &= \begin{bmatrix} R_X(1,1) & R_X(1,2) & \ldots & R_X(1,n) \\ R_X(2,1) & . & . & . \\ . & . & . & . \\ R_X(n,1) & . & . & R_X(n,n) \end{bmatrix} \end{aligned}$$

The covariance matrix is

$$\begin{aligned} \underline{K}_X &\stackrel{\text{def}}{=} \mathcal{E}\{[\underline{X}(u) - \underline{m}_X][\underline{X}(u) - \underline{m}_X]^{T*}\} \\ &= \mathcal{E}\{\underline{X}(u)\underline{X}^{*T}(u)\} - \underline{m}_X \mathcal{E}\{\underline{X}^{*T}(u)\} \\ &- \mathcal{E}\{\underline{X}(u)\}\underline{m}_X{}^{*T} + \underline{m}_X \underline{m}_X{}^{*T} \\ &\Rightarrow \boxed{\underline{K}_X = \underline{R}_X - \underline{m}_X \underline{m}_X{}^{*T}} \end{aligned}$$

2.2 Linear transformations - Spectral shaping and factorization

Suppose we are given a random vector $\underline{X}(u)$ and we construct another random vector $\underline{Y}(u)$ through the linear transformation

$$\begin{aligned} \underline{Y}(u) &= \underline{H}\underline{X}(u) \\ (m,1) &= (m,n)x(n,1) \end{aligned}$$

The question that we pose is " what is the second-moment description of $\underline{Y}(u)$? " We first look at the mean of $\underline{Y}(u)$

$$\underline{m}_Y = \mathcal{E}\{\underline{Y}(u)\} = \begin{bmatrix} \mathcal{E}\{Y(u,1)\} \\ \vdots \\ \mathcal{E}\{Y(u,m)\} \end{bmatrix} = \begin{bmatrix} h_{11} & \dots & h_{1n} \\ \vdots & & \vdots \\ h_{m1} & \dots & h_{mn} \end{bmatrix} \cdot \begin{bmatrix} \mathcal{E}\{X(u,1)\} \\ \vdots \\ \mathcal{E}\{X(u,n)\} \end{bmatrix}$$

$$\Rightarrow \boxed{\underline{m}_Y = \underline{H}\underline{m}_X}$$

In the above derivation we claimed that

$$\mathcal{E}\{\sum_{i=1}^{n} h_{ji}X(u,i)\} \stackrel{?}{=} \sum_{i=1}^{n} h_{ji}\mathcal{E}\{X(u,i)\}$$

In other words we assumed that expectation and summation can be interchanged but this holds only if $R_X(t,t)$ is finite (i.e. the variance is finite). For the autocorrelation function of Y :

$$\begin{aligned} \underline{R}_Y &= \mathcal{E}\{\underline{Y}(u)\underline{Y}^{*T}(u)\} \\ &= \mathcal{E}\{\underline{H}\underline{X}(u)(\underline{H}\underline{X}(u))^{*T}\} \\ &= \mathcal{E}\{\underline{H}\underline{X}(u)\underline{X}^{*T}(u)H^{*T}\} \\ &\Rightarrow \boxed{\underline{H}\ \underline{R}_X\ \underline{H}^{*T} = \underline{R}_Y} \end{aligned}$$

A useful concept is that a "white" vector, $\underline{W}(u)$, which is a randon vector with

$$\underline{m}_W = \underline{0}$$

and

$$\begin{aligned} \underline{R}_W &= \underline{K}_W = \sigma^2\underline{I} \\ &= \begin{bmatrix} \sigma^2 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma^2 \end{bmatrix} \end{aligned}$$

where σ is a constant and $\underline{I}$ is the identity matrix. This means that all components W_i of $\underline{W}$ are uncorrelated with all others, all have zero mean and variance σ^2.

There arises the question of *spectral shaping*: "Given a white vector $\underline{W}(u)$, can we find a linear transformation such that the resultant vector $\underline{X}(u)$ has a given mean $\underline{m}_X$ and a given covariance matrix $\underline{K}_X$? " We

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can answer this question by looking at our previous results. The mean is $\underline{m}_X = \underline{H}\underline{m}_W$ where $\underline{H}$ is the linear transformation (i.e. matrix) that we are looking for. Since $\underline{m}_W = \underline{0}$ it follows that $\underline{m}_X = \underline{0}$. Consequently

$$\begin{aligned} \underline{R}_X &= \underline{H}\underline{R}_W\underline{H}^{*T} \\ &= \sigma^2 \underline{H}\underline{I}\underline{H}^{*T} \end{aligned}$$

or

$$\boxed{\underline{R}_X = \sigma^2 \underline{H}\underline{H}^{*T}}$$

Therefore spectral shaping is equivalent to the following: Given a correlation matrix $\underline{R}_X$ find an $\underline{H}$ such that $\underline{R}_X = \underline{H}\underline{H}^{*T}$

Note :

σ^2 can be absorbed in the given $\underline{R}_X$ by creating a "new" given $\underline{R}_X$. Other names for this problem are "matrix factorization", "square root of a matrix"

Brief review of linear algebra

Definition 2.2.1 *A complex (real) matrix $\underline{A}$ is called* Hermitian symmetric (symmetric) *iff*

$$\underline{A} = \underline{A}^{*T}$$