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Chapter 1

INTRODUCTION

1.1 Review random variables

Recall the definition of a r.v. X(u) as a mapping from a probability space $(\mathcal{U}, \mathcal{F}, \mathcal{P})$ to the real line , where $\mathcal{U}$ is the space of all possible outcomes of an experiment , $\mathcal{F}$ is the Borel field on that space and $\mathcal{P}$ is the probability measure on $\mathcal{F}$.

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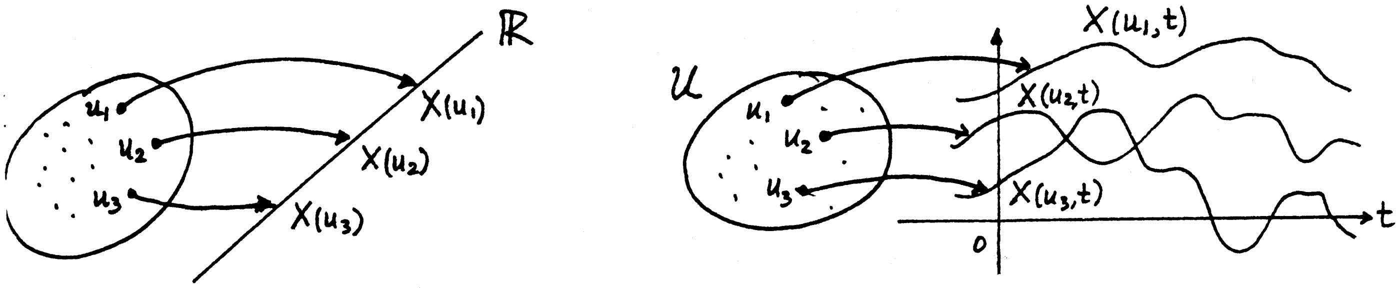
A random process is a mapping from a probability space to a set of functions.

1.2 Random processes - examples

We look at random processes (or stochastic processes) where the parameter space $t \in \mathcal{T}$ is discrete : $\ldots, -2, -1, 0, 1, 2, \ldots$ called a *random sequence.*



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Definition 1.2.1 *A stochastic process is a collection of random variables*

Examples of stochastic processes

1. Consider the free fall of a particle from some height H. If $h(t)$ is the distance travelled at time t and g is the gravitational constant then ideally

$$h(t) = H - \frac{1}{2}gt^2$$

Let T_h be the time it takes the particle to reach the ground or "hit" time. By solving $h(t) = 0$ we get

$$T_h = \sqrt{\frac{2H}{g}}$$

T_h is a random variable (r.v.) and $h(u,t)$ is a random process. It is a function of both $u \in \mathcal{U}$ and $t \in \mathcal{T}$.

If we fix the first argument $u = u_0$ then $h(u_0, t)$ is a deterministic function of t , called *sample path* If we fix time $t = t_0$ then $h(u, t_0)$ is a random variable.

2. The voltage of an FM receiver of a randomly chosen station is a random process.

3. The prices in the stock market form a random process.

Recall the definition of a random variable $X(u,t); u \in \mathcal{U}, t \in \mathcal{T}$. With respect to the choice of the parameter space $\mathcal{T}$ some cases of interest are :

1. for a finite set $\mathcal{T} = [1, 2, \ldots, n]$ the random process is just a collection of r.v.'s $X(u,1), \ldots, X(u,n)$ which is formulated as a random vector

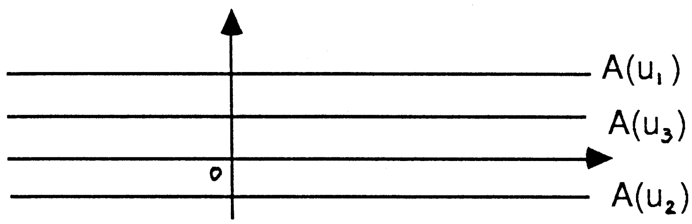
$$\underline{X}(u) = \begin{bmatrix} X(u,1) \\ \vdots \\ X(u,n) \end{bmatrix}$$

2. if $\mathcal{T}$ is a finite line segment

$$\mathcal{T} = \{t; 0 \leq t \leq T\}$$

then it is like observing a r.p. between time 0 and T.

3. Infinite line $\mathcal{T} \equiv \mathcal{R}$



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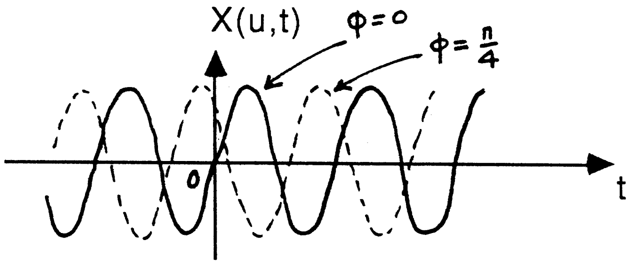

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4. Countable set

$$\mathcal{T} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

in which case we get a random sequence

Note : It is not necessary that parameter t signify "time" , it could be space. Looking at the range of $X(u,t)$ it can be the real line $\mathcal{R}$, or the space of complex numbers $\mathcal{C}$. Recall the definition of a comlpex r.v. :

$$Z(u) \stackrel{\text{def}}{=} X(u) + jY(u)$$

where X(u) and Y(u) are real r.v.'s

Examples

1. $X(u,t)$= A(u) , a random constant. For each outcome of the experiment we get a constant.

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2. Sine wave with random phase

$$X(u,t) = sin(2\pi ft + \phi(u))$$

where ϕ is a r.v. that could be uniformly distributed in (0,2π).

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Further modelling is achieved by having the amplitude of the sine wave be a r.v. A(u)

$$X(u,t) = A(u)sin(2\pi ft + \phi(u)$$

$F(x_1; t_1, x_2, t_2; \cdots x_n, t_n)$
$= Pr\{X(u, t_1) \le x_1,$
$X(u, t_2) \le x_2, \cdots$

3. A random walk

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4. We cannot describe the sample paths at all, neither in a mathematical nor visual way. Here we can only give statistical information about the process.

1.3 STATISTICAL DESCRIPTION

1.3.1 The most general kind of stochastic information for a process is obtained by the joint probability distribution function of a number n of samples $X(u,t_1), X(u,t_2), \ldots, X(u,t$ for any n and any set $\{t_1, t_2, \ldots, t_n\}$, defined as

~~Special~~ cases:
First-order;

$$F_X(x,t) \stackrel{\text{def}}{=} Pr\{X(u,t) \le x\}$$

or if this function is differentiable the joint density function

$$f_X(x,t) = \frac{\vartheta F_X(x,t)}{\vartheta x}$$

Second-order

1.3.2 Correlation and covariance functions

~~A more advanced description of a random process~~ involves second order statistics such as the joint PDF of two random variables obtained from the process by looking at time t_1 and t_2

$$F(x_1, x_2; t_1, t_2) = Pr\{X(u,t_1) \le x_1, X(u,t_2) \le x_2\}$$

If this function F is differentiable then we can define the joint pdf as

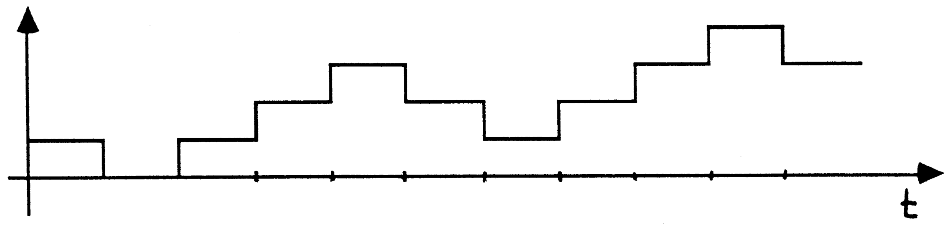
$$f(x_1, x_2; t_1, t_2) = \frac{\vartheta^2 F}{\vartheta x_1 \vartheta x_2}$$

This is typically too demanding so we usually settle for less. Moments are desirable quantities to obtain such as

1. The mean value of $X(u,t)$

$$m_X(t) \stackrel{\text{def}}{=} \mathcal{E}\{X(u,t)\}$$

which is a deterministic function of t



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2. The correlation function

$$R_X(t_1, t_2) \stackrel{\text{def}}{=} \mathcal{E}\{X(u, t_1)X(u, t_2)\}$$

which is again a deterministic function of two arguments t_1 and t_2. The above

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definition for the autocorrelation holds for real $X(u, t)$. If $X(u, t)$ is a complex process then we define

$$R_X(t_1, t_2) \stackrel{\text{def}}{=} \mathcal{E}\{X(u, t_1)X^*(u, t_2)\}$$

where * means comlex conjugate

3. The *covariance function*

$$K_X(t_1, t_2) \stackrel{\text{def}}{=} \mathcal{E}\{(X(u, t_1) - m_X(t_1))(X(u, t_2) - m_X(t_2))^*\}$$

by expanding we conclude that

$$K_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1)m_X(t_2)^*$$

Note : If $m_X(t) = 0$ then $K_x = R_x$ The above constitute second-order description of a random process, which very often is all we have or can calculate.
In general knowing $m_X(t)$ and $R_X(t_1, t_2)$ says nothing about the underlying statistics which generated them. A notable exception is the *Gaussian* case that we will see later.

Examples :

1. $X(u, t) = A(u)$

$$\Rightarrow \qquad m_X(t) = m_A = \mathcal{E}\{A(u)\}$$

Note : Suppose that $m_A = 0$, ie the *ensemble* average of $X(u, t)$ is zero. Yet every time we do the experiment, (with probability 1 for continuous rv.'s) we see a constant number $\neq 0$! (for $-\infty < t < \infty$) Here the sample paths have little relation to the statistical averages of the process. Processes for which the sample path behaviour relates to ensemble quantities are called *ergodic*. (we will discuss them later)