Discrete-Time random processes

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Review of C-T R.P.

- Second order statistical description of a *real* continuous-time (C-T) random process (R.P.) $X(t;u)$:
  - Mean: $m_X(t) \triangleq \mathcal{E}\{X(t;u)\}$
  - Auto-correlation: $R_X(t_1,t_2) \triangleq \mathcal{E}\{X(t_1)X(t_2)\}$

- For a WSS random process:

$$R_X(t_1,t_2) = R_X(t_1-t_2)$$

$$= R_X(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_X(f)$$

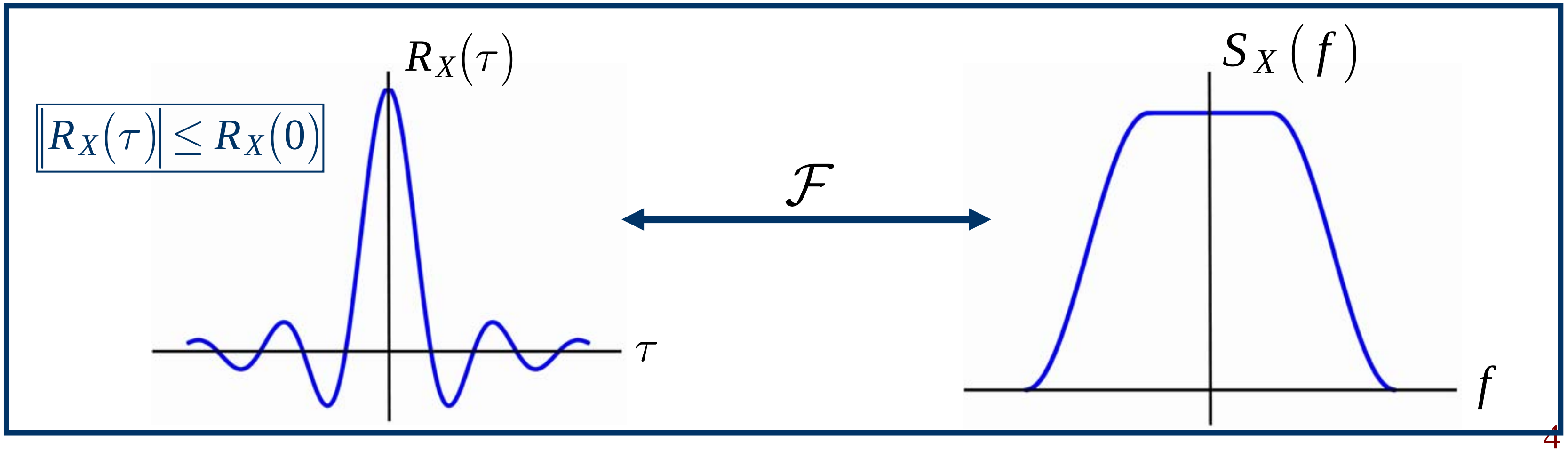
Properties of $R_X(\tau)$ and $S_X(f)$

- $R_X(\tau)$ is symmetric and positive definite

$$R_X(\tau) = R_X(-\tau); \qquad \sum_i \sum_j a_i a_j R_X(t_1 - t_2) \geq 0$$

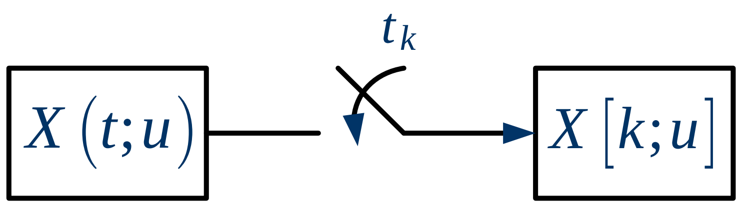
- $S_X(f)$ is symmetric and positive

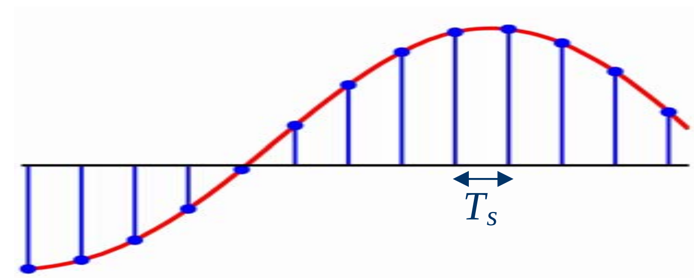
$$S_X(f) = S_X(-f); \qquad S_X(f) \geq 0$$



Generation of a D-T R.S. by Sampling

- A discrete-time (D-T) random ***sequence*** (R.S.) $\{X[k;u]\}$ can be generated by sampling a C-T R.P. $X(t;u)$ every T_s (i.e., at sampling instants $t_k = kT_s;\ \ k = 0, \pm 1, \pm 2, \dots$)





- The second order statistical description of $\{X[k;u]\}$ is:
  - $m[k] \triangleq \mathcal{E}\{X[k]\} = \mathcal{E}\{X(kT_s)\} = m_X(kT_s)$
  - $R_X[i,j] \triangleq \mathcal{E}\{x[i]x[j]\} = R_X(iT_s, jT_s)$

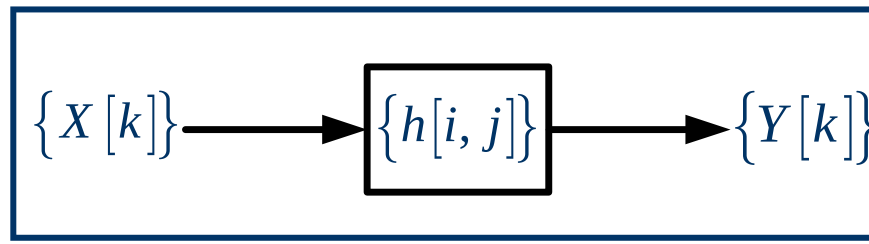
$$\overset{\text{W.S.S.}}{=} R_X((i-j)T_s)$$

$$\triangleq R_X[m] \qquad\qquad (m = i - j)$$

LTI Filtering of D-T R.S. (1/2)

- In general, a linear filtering operation on a D-T R.S. is described by:

$$Y[k;u] = \sum_{j=-\infty}^{\infty} h[k,j]\, X[j;u]$$

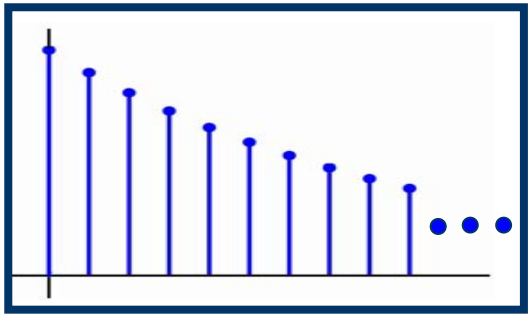


- If the system is LTI (i.e., $h[i,j] = h[i-j]$)

$$\begin{aligned} Y[k] &= \sum_{j=-\infty}^{\infty} h[k-j]\, X[j] \\ &\overset{n=k-j}{=} \sum_{n=-\infty}^{\infty} h[n]\, X[k-n] \\ &= h[k] * X[k] \end{aligned}$$

LTI Filtering of D-T R.S. (2/2)

- If the system is causal (i.e., $h[n]=0;\ n<0$)



$$Y[k]=\sum_{n=0}^{\infty} h[n]X[k-n]$$
$$=h[0]X[k]+h[1]X[k-1]+\cdots+h[l]X[k-l]+\cdots$$

- If the system is causal and finite (i.e., $h[n]=0;\ n\notin[0,L]$)

$$Y[k]=\sum_{n=0}^{L} h[n]X[k-n]$$

Second order statistics of the output R.S.

- Starting from $Y[k] = h[k] * X[k]$, it can be shown that
  - $m_Y[k] = h[k] * m_X[k]$
  - $R_Y[m] = h[m] * R_X[m] * h[-m]$

  or $\boxed{R_Y[m] = R_X[m] * r_h[m]}$

  where $r_h[m] \triangleq h[m] * h[-m] = \sum_{j=-\infty}^{\infty} h[j]h[-(m-j)]$

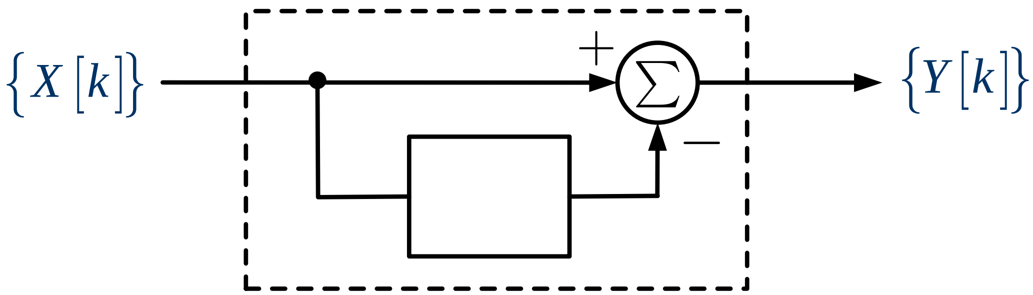
  $$= \sum_{j=-\infty}^{\infty} h[j]h[j-m]$$

  is the *deterministic* autocorrelation of $\{h[m]\}$

- Note that $r_h[0] = \sum_{j=-\infty}^{\infty} h^2[j] =$ energy of $\{h[k]\}$

  and $r_h[m] = r_h[-m]$

Example 1 (1/2)

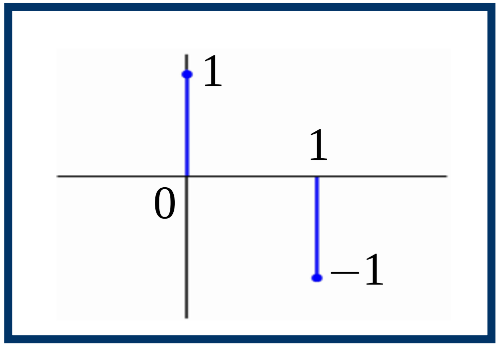


- The system is described by the differential equation

$$Y[k] = X[k] - X[k-1] \quad \forall k$$

- or by the impulse response

$$h[k] = \delta[k] - \delta[k-1] = \begin{cases} 1 \;; & k = 0 \\ -1 \;; & k = 1 \\ 0 \;; & \text{otherwise} \end{cases}$$

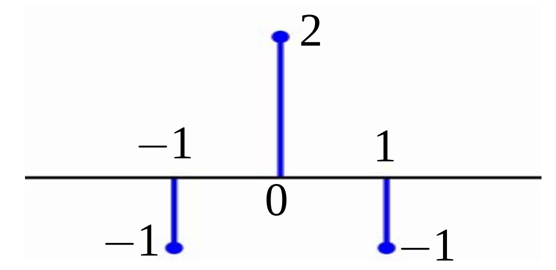


Example 1 (2/2)

- The deterministic autocorrelation of $\{h[m]\}$ is

$$\begin{aligned} r_h[m] &= h[m] * h[-m] \\ &= \left(\delta[m]-\delta[m-1]\right) * \left(\delta[m]-\delta[m+1]\right) \\ &= 2\delta[m]-\delta[m-1]-\delta[m+1] \end{aligned}$$

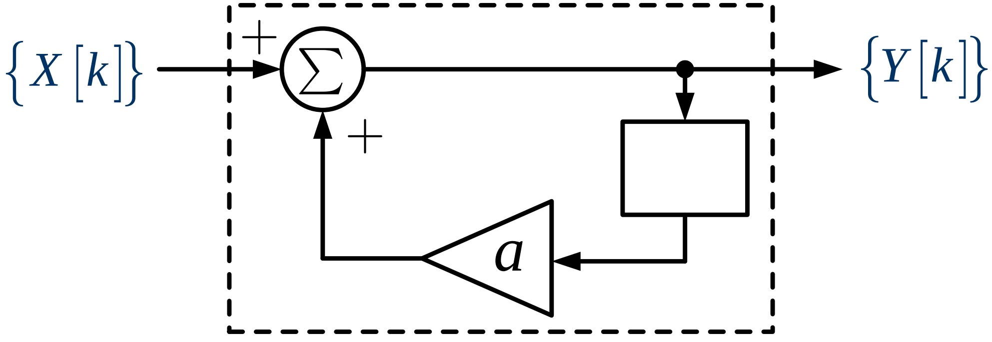
Recall that
$\delta[-k]=\delta[k]$



- Therefore, $R_Y[m]$ is

$$\begin{aligned} R_Y[m] &= R_X[m] * r_h[m] \\ &= 2R_X[m]-R_X[m-1]-R_X[m+1] \end{aligned}$$

Example 2 (1/2)

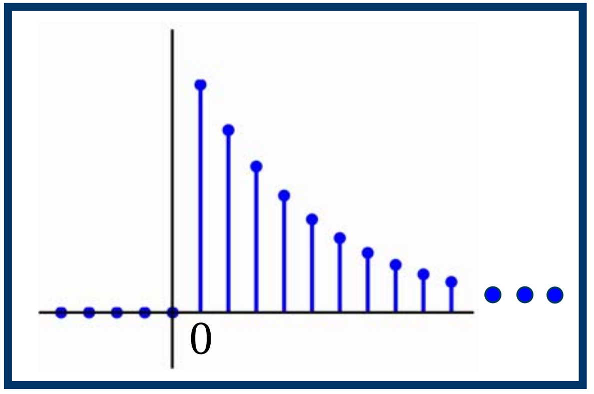


- Here, the equation describing the system is

$$Y[k] = aY[k-1] + X[k] \quad \forall k$$

- and the impulse response is

$$h[k] = \begin{cases} 0 & ; k < 0 \\ a^k & ; k \geq 0 \end{cases}$$

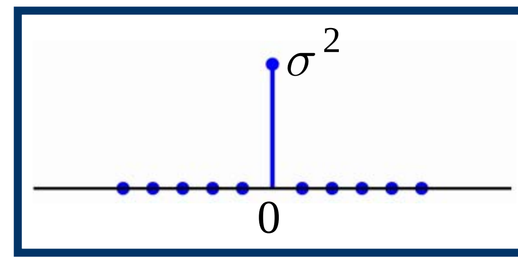


Example 2 (2/2)

- The deterministic autocorrelation is (prove it):

$$r_h[m] = a^{|m|} \Big/ \left(1 - a^2\right)$$

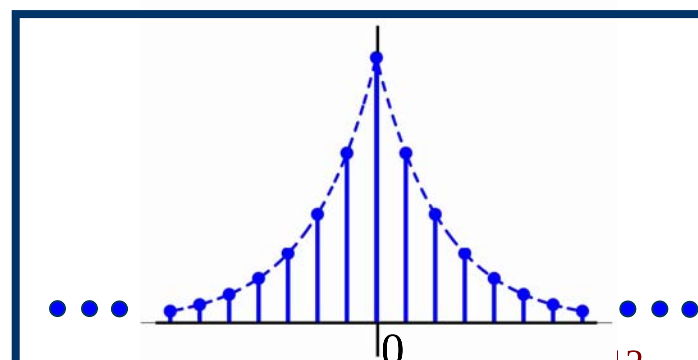
- If the input sequence $\{X[k]\}$ is white, i.e.,



$$R_X[m] = \begin{cases} \sigma^2 & ; m = 0 \\ 0 & ; m \neq 0 \end{cases}$$

then, the output sequence autocorrelation is

$$R_Y[m] = \frac{\sigma^2}{1 - a^2} a^{|m|}$$



Autoregressive processes

- The R.P. $\{Y[k]\}$ described by

$$Y[k]+a_1Y[k-1]+\cdots+a_MY[k-M]=W[k]$$

where $\{W[k]\}$ is a white sequence, is called an autoregressive process of order M.

Z-domain Description of WSS R.S.

- Given a WSS R.S. $\{X[k]\}$, we form the *Z*-transform $S_X(z)$ of the autocorrelation $R_X[m]$:

$$S_X(z) = \mathcal{Z}\{R_X[m]\} \triangleq \sum_{m=-\infty}^{\infty} R_X[m] z^{-m}$$

- The power spectrum of $\{X[k]\}$ is $S_X(e^{j\omega})$ (i.e., $S_X(z)$ evaluated on the unit circle)

- <u>Question</u>: What is the *Z*-domain relation between a system's input and output sequences

Z-Domain Filtering

- The impulse response has a *Z*-transform $\mathcal{Z}\{h[k]\} = H(z)$

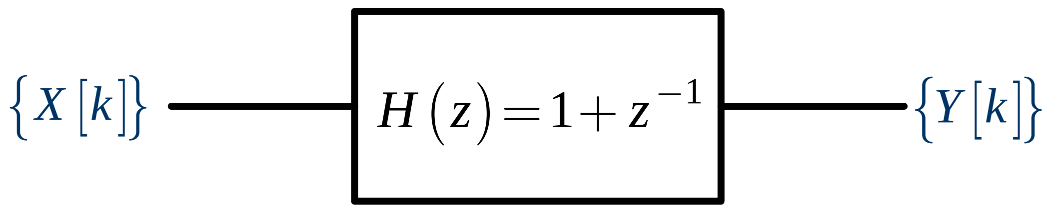
- The *Z*-transform of $r_h[m]$ is

$$\mathcal{Z}\{r_h[m]\} = \mathcal{Z}\{h[m] * h[-m]\} = \mathcal{Z}\{h[m]\} \cdot \mathcal{Z}\{h[-m]\}$$
$$= H(z) H(z^{-1}) \qquad (\text{real } \{h[k]\})$$

- Thus, applying the *Z*-transform on both sides of the equation $R_Y[m] = R_X[m] * r_h[m]$, we have

$$\mathcal{Z}\{R_Y[m]\} = \mathcal{Z}\{R_X[m]\} \mathcal{Z}\{r_h[m]\}$$
$$\Rightarrow \quad \boxed{S_Y(z) = S_X(z) H(z) H(z^{-1})}$$

Example 3



- We want to find $S_Y(z)$. The system shown above is similar to the one in example 1. It can be found that

$$R_Y[m] = 2R_X[m] + R_X[m-1] + R_X[m+1]$$

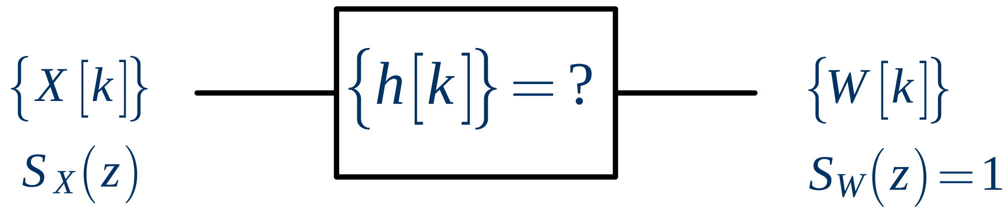
$$\stackrel{\mathcal{Z}}{\Rightarrow} S_Y(z) = 2S_X(z) + z^{-1}S_X(z) + zS_X(z)$$

- Another way to find $S_Y(z)$ would be as

$$\begin{aligned} S_Y(z) &= S_X(z)H(z)H\left(z^{-1}\right) \\ &= S_X(z)(1+z)\left(1+z^{-1}\right) \end{aligned}$$

which is of course, the same result

Example 4 (1/3)



- ***Whitening Problem***: What is the required impulse response $\{h[k]\}$ so that the output is white?

- *Solution*: $S_X(z)$ is factorized as $S_X(z)=G(z)G(z^{-1})$. We then have:

$$1=S_X(z)H(z)H(z^{-1})$$

$$1=G(z)G(z^{-1})H(z)H(z^{-1})$$

$$1=\big(G(z)H(z)\big)\big(G(z^{-1})H(z^{-1})\big) \Rightarrow \boxed{H(z)=\frac{1}{G(z)}}$$

Example 4 (2/3)

- Consider the case where $S_X(z) = 1/(2 + z + z^{-1})$. In order to find a whitening filter, $S_X(z)$ must be factorized. This can be accomplished as follows:

$$S_X(z) = \frac{1}{z^{-1}(2z + z^2 + 1)} = \frac{1}{z^{-1}(z+1)^2}$$

$$= \frac{1}{z^{-1}(z+1)(z+1)}$$

$$= \frac{1}{(1+z^{-1})(1+z)}$$

$$= \frac{1}{1+z^{-1}} \cdot \frac{1}{1+z}$$

Example 4 (3/3)

- There are two possible choices for $G(z)$ (and also for $H(z)$)

- For $G(z) = 1/(1+z^{-1}) \Rightarrow H(z) = 1/G(z) = 1+z^{-1}$

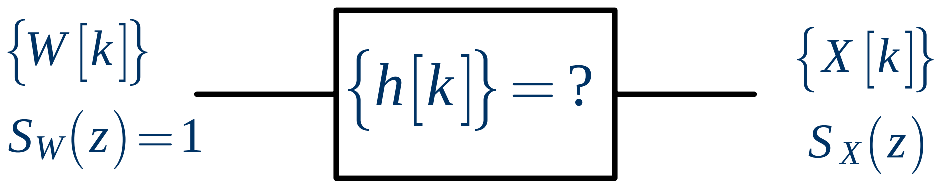
$$\Rightarrow h[k] = \delta[k] + \delta[k-1]$$

- For $G(z) = 1/(1+z) \Rightarrow H(z) = 1/G(z) = 1+z$

$$\Rightarrow h[k] = \delta[k] + \delta[k+1]$$

- Both choices are valid. The first filter is causal, whereas the second in non-causal.

Example 5



- ***Coloring Problem***: What is the required impulse response $\{h[k]\}$ so that for a white input the output has spectrum $S_X(z)$?

- *Solution*: $S_X(z)$ is factorized as $S_X(z)=G(z)G\left(z^{-1}\right)$. We then have: $S_X(z)=1\cdot H(z)H\left(z^{-1}\right)$

$$G(z)G\left(z^{-1}\right)=H(z)H\left(z^{-1}\right)$$

$$G(z)\underbrace{U(z)U\left(z^{-1}\right)}_{=1}G\left(z^{-1}\right)=H(z)H\left(z^{-1}\right)$$

- The required filter is $\boxed{H(z)=G(z)=U(z)G(z)}$