«Random Processes»

Lecture #2: Random Vectors

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Introduction

- **Contents:**
  - Definitions: Correlation and Covariance matrix
  - Linear transformations: Spectral shaping and factorization
  - The whitening concept
  - The Karhunen-Loeve expansion

Introduction

<u>Definition-Correlation and Covariance matrix:</u>

Random vectors come about either by 'sampling' one random process $X(u,t)$ at $t_1, t_2, \ldots, t_N$ or by 'observing' a number of processes $X_1(u,t), X_2(u,t), \ldots, X_N(u,t)$ at the same time. Essentially the two ways are equivalent mathematically.

$$\underline{X}(u) \triangleq \begin{bmatrix} X(u,1) \\ \vdots \\ X(u,N) \end{bmatrix}, \qquad \mathcal{E}\{\underline{X}(u)\} \triangleq \begin{bmatrix} m_X(1) \\ \vdots \\ m_X(N) \end{bmatrix}$$

Introduction

- The autocorrelation function is:

$$\underline{R}_X \triangleq \mathcal{E}\left\{\underline{X}(u)\,\underline{X}^{*T}(u)\right\}$$

$$= \mathcal{E}\left\{\begin{bmatrix} X(u,1) \\ \vdots \\ X(u,N) \end{bmatrix}\left[X^*(u,1),\ldots,X^*(u,N)\right]\right\}$$

$$= \begin{bmatrix} R_X(1,1) & R_X(1,2) & \cdots & R_X(1,N) \\ R_X(2,1) & R_X(2,2) & \cdots & R_X(2,N) \\ \vdots & \vdots & \vdots & \vdots \\ R_X(N,1) & R_X(N,2) & \cdots & R_X(N,N) \end{bmatrix}$$

Introduction

The covariance matrix is:

$$\underline{K}_X \triangleq \mathcal{E}\left\{\left(X(u)-\underline{m}_X\right)\left(X(u)-\underline{m}_X\right)^{*T}\right\}$$

$$=\mathcal{E}\left\{X(u)X^{*T}(u)\right\}-\underline{m}_X\mathcal{E}\left\{X^{*T}(u)\right\}-\mathcal{E}\left\{X^{*T}(u)\right\}\underline{m}_X^{*T}+\underline{m}_X\underline{m}_X^{*T}$$

$$\boxed{\Rightarrow \underline{K}_X=\underline{R}_X-\underline{m}_X\underline{m}_X^{*T}}$$

Linear transformations

Suppose we are given a random vector $\underline{X}(u)$ and we construct another random vector $\underline{Y}(u)$ through the linear transformation

$$\underline{Y}(u) = \underline{H}\underline{X}(u)$$

$$\left(y_m = \sum_{n=1}^{N} h_{mn} x_n ;\ m = 1, 2, \ldots, M \right)$$

<u>Question</u>: What is the second-moment description of $\underline{Y}(u)$?

Linear transformations

$$\underline{m}_Y = \mathcal{E}\left\{\underline{Y}(u)\right\} = \begin{bmatrix} \mathcal{E}\left\{Y(u,1)\right\} \\ \vdots \\ \mathcal{E}\left\{Y(u,M)\right\} \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{1N} \\ \vdots & & \vdots \\ h_{M1} & \cdots & h_{MN} \end{bmatrix} \cdot \begin{bmatrix} \mathcal{E}\left\{X(u,1)\right\} \\ \vdots \\ \mathcal{E}\left\{X(u,N)\right\} \end{bmatrix}$$

$$\boxed{\Rightarrow \underline{m}_Y = \underline{H}\,\underline{m}_X}$$

- In the above derivation we claimed that:

$$\mathcal{E}\left\{\sum_{n=1}^{N} h_{mn} X(u,n)\right\} \stackrel{?}{=} \sum_{n=1}^{N} h_{mn}\, \mathcal{E}\left\{X(u,n)\right\}$$

In others words we assumed that expectation and summation can be interchanged, but this holds only if $R_X(t_{1,}, t_2)$ is finite.

Linear transformations

For the autocorrelation function of $\underline{Y}(u)$:

$$\begin{aligned}
\underline{R}_Y &= \mathcal{E}\left\{\underline{Y}(u)\underline{Y}^{*T}(u)\right\} \\
&= \mathcal{E}\left\{\left(\underline{H}\underline{X}(u)\right)\left(\underline{H}\underline{X}(u)\right)^{*T}\right\} \\
&= \mathcal{E}\left\{\underline{H}\underline{X}(u)\underline{X}^{*T}(u)\underline{H}^{*T}\right\} \\
&= \underline{H}\mathcal{E}\left\{\underline{X}(u)\underline{X}^{*T}(u)\right\}\underline{H}^{*T}
\end{aligned}$$

$$\boxed{\Rightarrow \underline{R}_Y = \underline{H}\underline{R}_X\underline{H}^{*T}}$$

“White” vectors

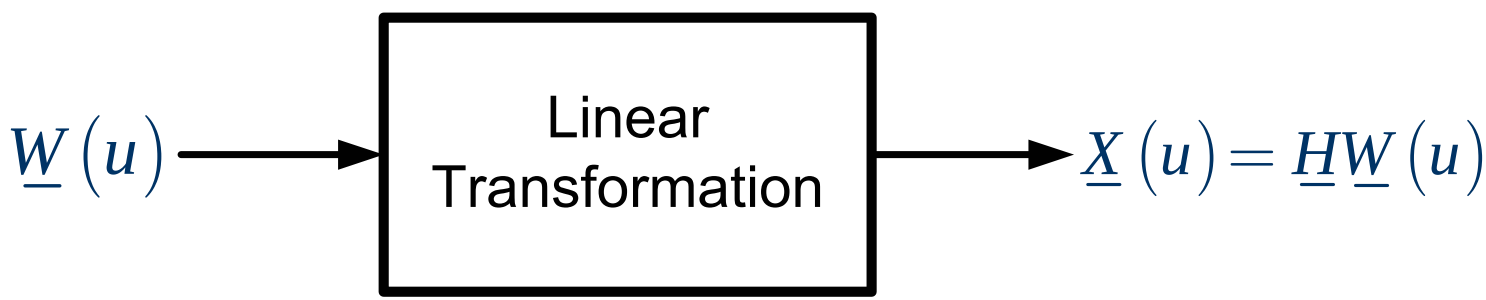
A useful concept is that of a “white” vector $\underline{W}(u)$, which is a random vector with mean $\underline{m}_W = \underline{0}$, and covariance matrix:

$$\underline{R}_W = \underline{K}_W = \sigma^2 \underline{I} = \begin{bmatrix} \sigma^2 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma^2 \end{bmatrix}$$

where σ is a constant and $\underline{I}$ is the identity matrix. This means that all components w_i of $\underline{W}(u)$ are uncorrelated with each other, with zero mean and variance σ^2.

Spectral Shaping

- **<u>Problem</u>**: *Given the white vector* $\underline{W}(u)$, *can we find a linear transformation such that the resultant vector* $\underline{X}(u) = \underline{H}\underline{W}(u)$ *has given mean* $\underline{m}_X$ *and given covariance matrix* $\underline{K}_X$?



Spectral Shaping

- Since, $\underline{m}_W = \underline{0}$, it follows that

$$\underline{m}_X = \underline{H}\underline{m}_W = \underline{0}$$

- The covariance matrix of $\underline{X}(u)$ is:

$$\begin{aligned} \underline{R}_X &= \underline{H}\underline{R}_W \underline{H}^{*T} \\ &= \sigma^2 \underline{H}\underline{H}^{*T} \end{aligned}$$

- Therefore, spectral shaping is equivalent to the following:

Given a correlation matrix $\underline{R}_X$, *find an* $\underline{H}$ *such that* $\underline{R}_X = \underline{H}\underline{H}^{*T}$

Note: σ^2 can be absorbed in the given $\underline{R}_X$ by creating a "new" given $\underline{R}'_X$.
Other names for this problem are "matrix factorization", "square root of a matrix"

Spectral Shaping

Definition: A complex (real) matrix $\underline{A}$ is called Hermitian symmetric iff:

$$\underline{A} = \underline{A}^{*T}$$

Definition: A complex (real) matrix $\underline{A}$ is called Unitary (orthogonal) iff:

$$\underline{A}\underline{A}^{*T} = \underline{I}$$

Spectral Shaping

Theorem: if $\underline{K}$ is Hermitian symmetric then there exists a unitary matrix $\underline{E}$ such that

$$\underline{K} = \underline{E}\,\underline{\Lambda}\,\underline{E}^{*T} \qquad \underline{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}$$

with $\lambda_n; n = 1, 2, \ldots, N,$ the eigenvalues of $\underline{K}$ (not necessarily distinct). In other words:

Hermitian symmetric matrices are always diagonalizable.

Theorem: A necessary and sufficient condition for such a $\underline{K}$ to be nonnegative definite is that $\lambda_n \geq 0; n = 1, 2, \ldots, N.$

Spectral Shaping

Theorem: Let $\underline{K}$ be Hermitian symmetric. Then for each distinct (simple) eigenvalue there corresponds an eigenvector which is orthogonal (orthonormal) to all others .To each eigenvalue of multiplicity k there correspond k linearly independent eigenvectors, which are orthogonal to all eigenvectors of the rest eigenvalues. These k eigenvectors can be made orthogonal by application of the Gram-Schmidt procedure

In summary, every Hermitian ($N \times N$) matrix has N orthonormal eigenvectors $\left\{\underline{e}_n\right\}_{n=1}^{N}$, associated with its N eigenvalues $\left\{\lambda_n\right\}_{n=1}^{N}$. In fact, matrix $\underline{E}$ consists of these $\underline{e}_n$`s as its columns, i.e.,

$$\underline{E} = \left[\, \underline{e}_1 \mid \underline{e}_2 \mid \cdots \mid \underline{e}_N \,\right]$$

Spectral Shaping

Returning to the factorization problem, we want to find an $\underline{H}$ such that $\underline{R}_X = \underline{H}\underline{H}^{*T}$. Writing $\underline{R}_X = \underline{E}\underline{\Lambda}\underline{E}^{*T}$ (since $\underline{R}_X$ is Hermitian) we have

$$
\begin{aligned}
\underline{R}_X &= \underline{E}\underline{\Lambda}\underline{E}^{*T} \\
&= \underline{E}\underline{\Lambda}^{1/2}\underline{\Lambda}^{1/2}\underline{E}^{*T} \\
&= \underline{E}\underline{\Lambda}^{1/2}\left(\underline{\Lambda}^{1/2}\right)^{*T}\underline{E}^{*T} \quad ; \quad \underline{\Lambda}^{1/2} \triangleq \begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_N} \end{bmatrix} \\
&= \underbrace{\left(\underline{E}\underline{\Lambda}^{1/2}\right)}_{\underline{H}}\underbrace{\left(\underline{E}\underline{\Lambda}^{1/2}\right)^{*T}}_{\underline{H}^{*T}}
\end{aligned}
$$

Spectral Shaping

- We have arrived at a solution where

$$\boxed{\underline{H} = \underline{E}\,\underline{\Lambda}^{1/2}}$$

- However this solution is not unique. To see this, take any unitary matrix $\underline{U}$ and observe that:

$$R_X = \underline{H}\,\underline{H}^{*T} = \left(\underline{E}\,\underline{\Lambda}^{1/2}\right)\left(\underline{E}\,\underline{\Lambda}^{1/2}\right)^{*T}$$

$$= \left(\underline{E}\,\underline{\Lambda}^{1/2}\right)\underline{U}\,\underline{U}^{*T}\left(\underline{E}\,\underline{\Lambda}^{1/2}\right)^{*T}$$

$$= \underbrace{\left(\underline{E}\,\underline{\Lambda}^{1/2}\underline{U}\right)}_{\text{another } \underline{H}}\left(\underline{E}\,\underline{\Lambda}^{1/2}\underline{U}\right)^{*T}$$

Spectral Shaping

- Sometimes we take $U = E^{*T}$ and the resulting $\underline{H}$ is given as:

$$\underline{H} = \underline{E}\underline{\Lambda}^{1/2}\underline{U} = \underline{E}\underline{\Lambda}^{1/2}\underline{E}^{*T}$$

This matrix is often called the "square root" of $\underline{R}_X$

- From an applications viewpoint this factorization is useful in simulation, i.e., creating a random vector with desired correlation properties, starting from a "random number generator".

- **Note**: if $\underline{m}_X \neq 0$ then the appropriate linear transformation is:

$$\underline{X} = \underline{H}\underline{W} + \underline{m}_X$$

where the factorization is done on $\underline{K}_X$, *not on* $\underline{R}_X$.

Spectral Shaping

- **Example:**

  The required covariance matrix is:

$$\underline{K}_X = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}$$

The eigenvalues are found by solving the characteristic equation:

$$\det\left\{\underline{K}_X - \lambda_n \underline{I}\right\} = 0;\ \ n = 1,2,3$$

$$\Rightarrow \lambda_1 = 0,\ \lambda_2 = \lambda_3 = 3/2$$

Spectral Shaping

Solving for the corresponding eigenvectors we get:

$$\lambda_1 = 0 \Rightarrow \underline{e}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\lambda_2 = \frac{3}{2} \Rightarrow \underline{e}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\underline{e}_3 = \sqrt{\frac{2}{3}}\begin{bmatrix}1/2\\1/2\\-1\end{bmatrix}$$

Spectral Shaping

Therefore, we could choose the linear transformation:

$$\underline{H} = \underline{E}\,\underline{\Lambda}^{1/2} = \left[\, \underline{e}_1 \mid \underline{e}_2 \mid \underline{e}_3 \,\right] \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{3}/2 & 1/2 \\ 0 & -\sqrt{3}/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\underline{X} = \underline{H}\,\underline{W} = \begin{bmatrix} 0 & \sqrt{3}/2 & 1/2 \\ 0 & -\sqrt{3}/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} W(u,1) \\ W(u,2) \\ W(u,3) \end{bmatrix}$$

- Notice that $\underline{X}$ does not depend on $W(u,1)$.

Spectral Shaping

- In the above, we solved the problem of spectral shaping which is equivalent to a covariance matrix factorization. The solution was unconstrained, i.e., we imposed no restrictions on the nature of the linear transformation $\underline{H}$

- Now assume that we impose the constraint of the linear transformation being causal.

Spectral Shaping

- **Definition**: A causal linear transformation is equivalent to $\underline{H}$ being lower triangular, i.e., the wanted linear transformation is

$$\begin{bmatrix} X(u,1) \\ \vdots \\ X(u,N) \end{bmatrix} = \begin{bmatrix} h_{11} & 0 & \cdots & 0 \\ h_{21} & h_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ h_{N1} & h_{N2} & \cdots & h_{NN} \end{bmatrix} \cdot \begin{bmatrix} W(u,1) \\ \vdots \\ W(u,N) \end{bmatrix} + \underline{m}_X$$

$$\left(X(u,n) = \sum_{l=1}^{n} h_{nl} W(u,l);\ n = 1,2,\ldots,N \right)$$

- The problem can now be restated as

> *Find a lower-triangular matrix* $\underline{H}$ *such that:* $\underline{K}_X = \underline{H}\underline{H}^{*T}$

- Note: This factorization is called the "Cholesky factorization" of positive definite matrices

Spectral Shaping

- **Example (real-valued covariance matrix):**

$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ k_{N1} & k_{N2} & \vdots & k_{NN} \end{bmatrix} = \begin{bmatrix} h_{11} & 0 & \cdots & 0 \\ h_{21} & h_{22} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NN} \end{bmatrix} \cdot \begin{bmatrix} h_{11} & h_{21} & \cdots & h_{N1} \\ 0 & h_{22} & \cdots & h_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{NN} \end{bmatrix}$$

$$k_{11} = h_{11}^2 \Rightarrow h_{11} = \pm\sqrt{k_{11}}$$

$$k_{12} = h_{11}h_{21} \Rightarrow h_{21} = k_{12}/h_{11}$$

$$\vdots$$

in the same manner we can find the rest of h_{ij} .

Properties - Spectral Resolution

- Assume a real covariance matrix $\underline{K}_X$. We can rewrite the factorization $\underline{K}_X = \underline{E}\,\underline{\Lambda}\,\underline{E}^T$ as

$$\underline{K}_X = \left[\lambda_1 \underline{e}_1 \mid \lambda_2 \underline{e}_2 \mid \cdots \mid \lambda_N \underline{e}_N\right] \begin{bmatrix} \underline{e}_1^T \\ \underline{e}_2^T \\ \vdots \\ \underline{e}_N^T \end{bmatrix}$$

or

$$\underline{K}_X = \sum_{n=1}^{N} \lambda_n \underline{e}_n \underline{e}_n^T$$

This shows that $\underline{K}_X$ can be decomposed (resolved) into a sum of N matrices, each of the form $\underline{e}_n \underline{e}_n^T$ with weight λ_n .

- The set of N eigenvectors $\{e_n\}_{n=1}^{N}$ constitutes a basis for the N-dimensional vector space

Properties - Spectral Resolution

- Every deterministic vector $\underline{A}$ can be expanded into a series

$$\underline{A} = \sum_{n=1}^{N} a_n \underline{e}_n$$

where $a_n = \langle \underline{A}, \underline{e}_n \rangle = \underline{A}^T \underline{e}_n$ is the projection of $\underline{A}$ on the basis vector $\underline{e}_n$

- Thus, vector $\underline{A}$ can be described in terms of its "projections" $\{a_n\}$ along the $\{\underline{e}_n\}_{n=1}^{N}$ coordinates

Properties - Spectral Resolution

- It is clear that we can create random vectors by choosing these projections as random variables $\{A_n(u)\}$, i.e.,

$$\underline{A} = \sum_{n=1}^{N} A_n(u)\,\underline{e}_n$$

- Note: If the eigenvectors have the form

$$\underline{e}_n = [0, 0, \ldots, 0, 1, 0, 0, \ldots, 0]^T$$

with 1 in the *n*-th position, then

$$\underline{A} = \begin{bmatrix} A_1(u) \\ \vdots \\ A_N(u) \end{bmatrix}$$

Properties - Directional Preference

- Suppose we are given the covariance matrix $\underline{K}_X$ of some vector $\underline{X}(u)$ and would like to project this vector on some unit-length vector $\underline{b}$ ($\sum_{n=1}^{N} b_n^2 = 1$). The projection is the inner product:

$$Y(u) = \langle \underline{X}(u), \underline{b} \rangle = \underline{X}^T(u)\underline{b}$$

- Assuming that $\underline{m}_X = \underline{0}$, the variance of $Y(u)$ equals

$$\text{var}\{Y(u)\} = \sigma_Y^2 = \mathcal{E}\{Y^2(u)\} = \mathcal{E}\{Y(u)Y(u)\} = \mathcal{E}\{\underline{b}^T\underline{X}(u)\underline{X}^T(u)\underline{b}\}$$

$$= \underline{b}^T\underline{K}_X\underline{b}$$

i.e., the variance of $Y(u)$ is a quadratic functional of the $\{b_n\}$ `s

Properties - Directional Preference

- "Directional preference" translates to finding those directions $\underline{b}$ where the variance $\sigma_Y^2 = \underline{b}^T \underline{K}_X \underline{b}$ is highest (or lowest). This is an optimization problem where we want to maximize the above quadratic form, subject to the unit-norm constraint. To solve this, we expand $\underline{b}$ on the orthonormal basis $\{\underline{e}_n\}_{n=1}^{N}$, i.e.,

$$\underline{b} = \sum_{n=1}^{N} b_n \underline{e}_n$$

so that $\sum_{n=1}^{N} b_n^2 = 1$. The quadratic form can now be written as:

$$\sigma_Y^2 = b^T K_X b = \left(\sum_{n=1}^{N} b_n e_n\right)^T K_X \left(\sum_{m=1}^{N} b_m e_m\right)$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{N} b_n b_m e_n^T K_X e_m$$

Properties - Directional Preference

- Recalling that $\underline{K}_X \underline{e}_m = \lambda_m \underline{e}_m$, σ_Y^2 can be written as

$$\sigma_Y^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} b_n b_m e_n^T \lambda_m e_m = \sum_{n=1}^{N} \sum_{m=1}^{N} \lambda_m b_n b_m \underbrace{e_n^T e_m}_{\delta_{nm}}$$

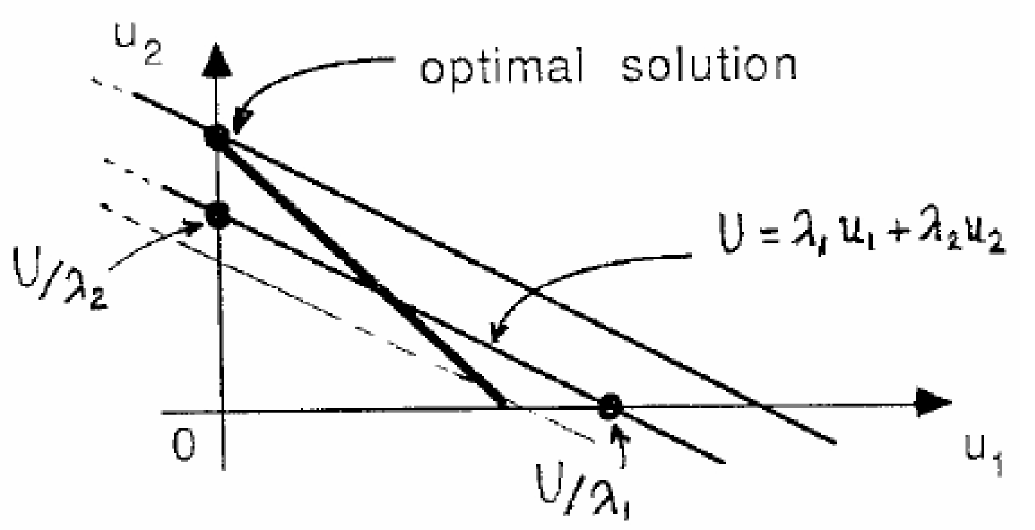
$$\Rightarrow \sigma_Y^2 = \sum_{n=1}^{N} \lambda_n b_n^2$$

- Now the original problem can be equivalently stated as follows:

Let $\{u_n\} \triangleq \{b_n^2\}$. *We want to maximize* $U = \sum_{n=1}^{N} \lambda_n u_n$
subject to the constraint $\sum_{n=1}^{N} u_n = 1$ *and* $u_i \geq 0,\ \lambda_i \geq 0$

Properties - Directional Preference

- **Example (*N* = 2)**:



For $\lambda_2 > \lambda_1$ the optimal solution is $u_1 = 0, u_2 = 1$. The general solution is to choose $u_m = 1$ where $\lambda_m = \max\{\lambda_n\}$ and $u_n = 0$ for $n \neq m$. Since $b_i^2 = 1; i = 1, 2, \ldots, N$, it follows that:

$$b_m = \pm 1$$

$$b_n = 0; n \neq m$$

Properties - Directional Preference

- The resulting variance is the maximum eigenvalue

$$\sigma_Y^2 = \lambda_m = \max\{\lambda_n\}$$

- Recalling that $\underline{b} = \sum_{n=1}^{N} b_n \underline{e}_n$ it follows that

$$\boxed{\underline{b}_{\max} = \underline{e}_{\max}}$$

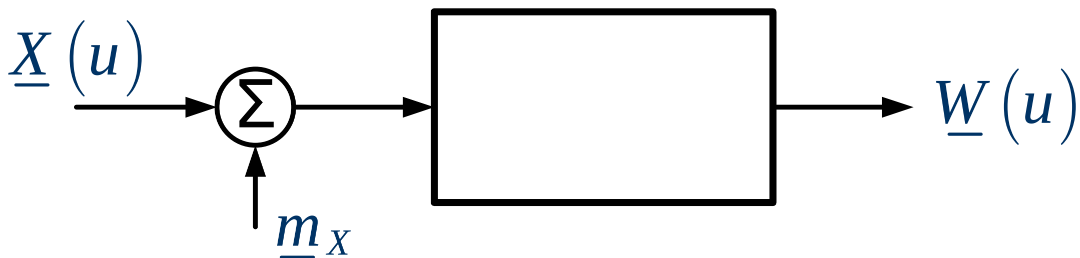
where $\underline{e}_{\max}$ is the eigenvector of $\underline{K}_X$ corresponding to the largest eigenvalue

- <u>Question</u>: What is the direction that *minimizes* the variance?

The whitening concept

- Converse to the factorization or spectral shaping problem
- **Problem statement**: Given a random vector $\underline{X}(u)$ with some mean $\underline{m}_X$ and covariance $\underline{K}_X$, find a linear transformation $\underline{G}$ such that the output $\underline{W}(u)$ is a white vector
  - $\underline{m}_X = 0$

  - $\underline{m}_X \neq 0$



The whitening concept

- From previous theory we know, that the covariance matrix of the "output vector" $\underline{W}(u)$ is

$$\underline{K}_W = \underline{G}\underline{K}_X\underline{G}^T$$

- For $\underline{W}(u)$ to be white we require $\underline{K}_W = \underline{I}$
- We also know that $\underline{K}_X$ can be factorized as (assuming real matrices)

$$\underline{K}_X = \underline{H}\underline{H}^T$$

- Thus, we require the following equality to hold:

$$\underline{G}\underline{H}\underline{H}^T\underline{G}^T = \underline{I}$$

$$\Rightarrow (\underline{G}\underline{H})(\underline{G}\underline{H})^T = \underline{I}$$

The whitening concept

- The simplest form of $\underline{G}$ that satisfies this equality is

$$\underline{G} = \underline{H}^{-1}$$

- However, since $\underline{H} = \underline{E}\underline{\Lambda}^{1/2}\underline{U}$, we can express $\underline{G}$ in terms of $\underline{E}$ and $\underline{\Lambda}$ as

$$\underline{G} = \underline{H}^{-1} = \left(\underline{E}\underline{\Lambda}^{1/2}\underline{U}\right)^{-1}$$
$$= \underline{U}^{-1}\underline{\Lambda}^{-1/2}\underline{E}^{-1}$$

- Recalling that $\underline{U}$ is by definition an arbitrary unitary matrix and $\underline{E}$ is also unitary since its columns are the orthonormal eigenvectors of $\underline{K}_X$, we end up at

$$\boxed{\underline{G} = \underline{U}^T\underline{\Lambda}^{-1/2}\underline{E}^T}$$

The Karhunen-Loeve expansion

- Starting from the coloring problem equation, we define the following random vectors

$$\underline{X}(u) = \underline{E}\,\underline{\Lambda}^{1/2} \underbrace{\underbrace{\underline{U}\,\underline{W}(u)}_{\underline{Y}(u) \triangleq \underline{U}\,\underline{W}(u)}}_{\underline{Z}(u) \triangleq \underline{\Lambda}^{1/2}\underline{Y}(u)}$$

- Claim: Vector $Y(u) = UW(u)$ is also white, and vector $Z(u) = \Lambda^{-1/2}Y(u)$ has uncorrelated components, each with a different variance

The Karhunen-Loeve expansion

- Proof: Using the standard formulas we obtain

$$\left.\begin{aligned} \underline{m}_Y &= \underline{U}\,\underline{m}_X = \underline{0} \\ \underline{K}_Y &= \underline{U}\,\underline{K}_W\underline{U}^T = \underline{U}\,\underline{U}^T = \underline{I} \end{aligned}\right\} \Rightarrow \underline{Y}(u) \text{ is white}$$

$$\left.\begin{aligned} \underline{m}_Z &= \underline{\Lambda}^{1/2}\underline{m}_Y = \underline{0} \\ \underline{K}_Z &= \underline{\Lambda}^{1/2}\underline{K}_Y\left(\underline{\Lambda}^{1/2}\right)^T \\ &= \underline{\Lambda}^{1/2}\underline{I}\,\underline{\Lambda}^{1/2} = \underline{\Lambda} \\ &= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix} \end{aligned}\right\} \Rightarrow \operatorname{var}\left\{Z_n(u)\right\} = \lambda_n \ ; \ n = 1, 2, \ldots, N$$

The Karhunen-Loeve expansion

- Rewriting the coloring problem equation as $\underline{X}(u) = \underline{E}\underline{Z}(u)$ we have:

$$\underline{X}(u) = \left[\underline{e}_1 \mid \underline{e}_2 \mid \cdots \mid \underline{e}_N\right] \begin{bmatrix} Z_1(u) \\ \vdots \\ Z_N(u) \end{bmatrix}$$

$$\Rightarrow \underline{X}(u) = \sum_{n=1}^{N} Z_n(u)\,\underline{e}_n$$

$$\Rightarrow \boxed{\underline{X}(u) = \sum_{n=1}^{N} \sqrt{\lambda_n} W_n(u)\,\underline{e}_n}$$

- This is the Karhunen – Loeve expansion of $\underline{X}(u)$. It states that every random vector can be written as a sum of orthonormal eigenvectors $\{\underline{e}_n\}$, each weighted by a random variable $W_n(u)$ and further scaled by $\sqrt{\lambda_n}$

The Karhunen-Loeve expansion

- Note that the Karhunen – Loeve expansion of a random vector $\underline{X}(u)$ is simply an expansion on a certain basis ($\{\underline{e}_n\}$) of the N-dimensional vector space. However, the basis is special, since (as we just showed) the projections $\langle \underline{X}(u), \underline{e}_n \rangle$ are uncorrelated random variables with variance λ_n . (Projecting on an arbitrary basis, would not have the same effect)

- One could say that a random vector has preferences into how it is going to be distributed in space!