«Random Processes»

Lecture #1: Introductory Concepts

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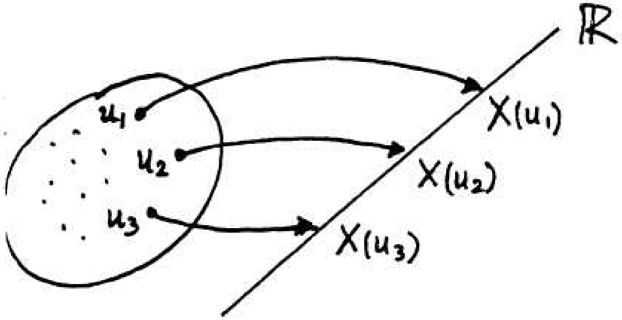
L_1: Contents

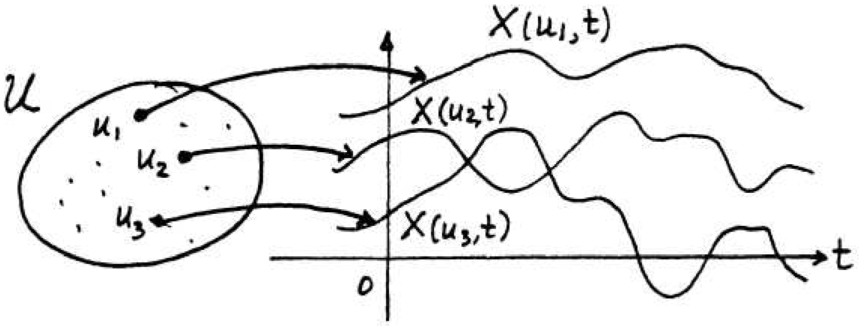
- **Introduction**
  - Review random variables
  - Random processes - Examples
  - Correlation and covariance functions

L_1: PV's versus RP's

Random Variable (RV):

Recall the definition of a r.v. X(u): it is a mapping from a probability space *{U,F,V)* to the real line , where *U* is the space of all possible outcomes of an experiment, *F* is a Borel field on that space and *V* is a probability measure on *F*.





Random Process (RP): it is a mapping from a probability space *{U,F,V)* to a set of functions

L_1: RP's versus RS's versus RVe's

Conclusion: *A stochastic process is a collection of random variables* $X(u,t_1)$, $X(u,t_2)$, …, $X(u,t_n)$,…

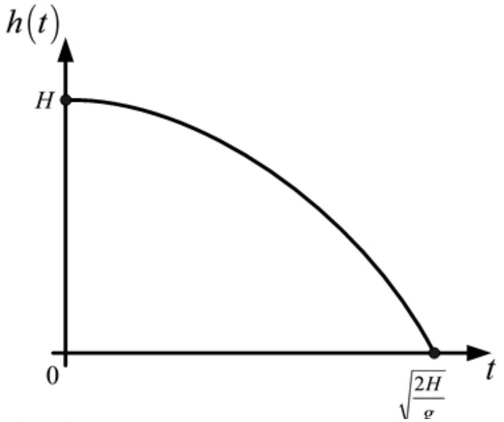
Note 1: A random process (or stochastic process) where the parameter space $t \in T$ is discrete : ..., -2, -1, 0, 1, 2,... called a *Random Sequence (RS).*

Note 2: If the parameter space is discrete and finite, $T=\{1,2,\ldots,N\}$, we have a *Random Vector* (RV) of dim (N,1)

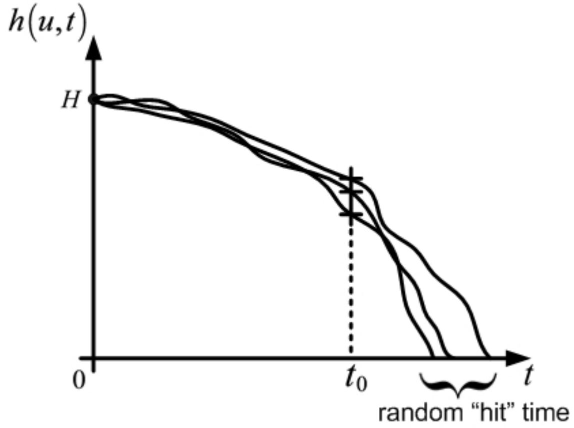
L_1: RP example #1

Consider the free fall of a particle from some height *H*. If *h(t)* is the distance traveled at time *t* and *g* is the gravitational constant then, ideally, *h(t)* is the deterministic function of t:

$$h(t) = H - \frac{1}{2} g t^2$$



However, with air turbulence,



L_1: RP example #1 (cont.)

Let T_h be the time it takes the particle to reach the ground or "hit" time. By solving $h(t) = 0$ we get:

$$T_h = \sqrt{\frac{2H}{g}}$$

However, in turbulence, T_h is a random variable (r.v.) and $h(u,t)$ is a random process. It is a function of both $u \varepsilon U$ and $t \varepsilon$ T.

• When we fix the first argument $u = u_0$ then $h(u_o,t)$ is a deterministic function of *t*, called *sample path*.

•When we fix time $t = t_o$, then $h(u,t_o)$ is a random variable.

∈

L_1: Other examples

- The voltage of an FM receiver of a randomly chosen station is a random process (explain the “double” randomness).
- The stock prices of a company form a random sequence.

L_1: Summary on RP cases

In general, we write $X(u,t); u \in \mathcal{U}, t \in \mathcal{T}$.
With respect to the choice of the parameter space *T*, cases of interest are :

1. A finite set *T = [1, 2,... ,N];* the random process is just a collection of r.v.'s *X(u, 1),... ,X(u,N)*, a random vector:

$$\underline{X}(u) = \begin{bmatrix} X(u,1) \\ \vdots \\ X(u,N) \end{bmatrix}$$

2. if T is an infinite (countable) series of integers, we have a RS

3. if T is a finite line segment $\mathcal{T} = \{t; 0 \leq t \leq T\}$
   then we "observe" an RP between time 0 and T.

4. Infinite line *T ≡R* is the continuous-time "classic" case

L_1: Summary on RP cases (2)

Note: It is not necessary that parameter *t* signify "time", it could be space ("random images") or anything else.

As for the range of *X{u,t)*, it can be the real line *R* , or the space of complex numbers *C*.

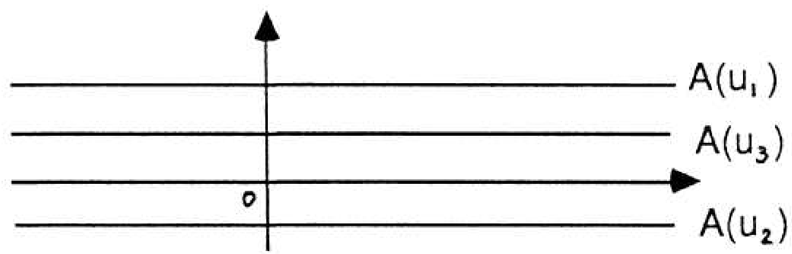
Recall the definition of a complex RV:

$$Z(u) \stackrel{\text{def}}{=} X(u) + jY(u)$$

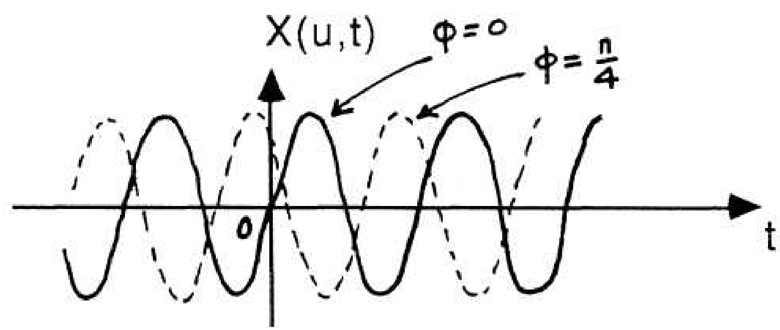
where X(u) and Y(u) are real r.v.'s

L_1: Classic cases of RP's

1. $X(u,t)$= A(u) , a random constant. For each outcome of the experiment we get a constant.



2. Sinewave with random phase: $X(u,t) = sin(2\pi f t + \phi(u))$



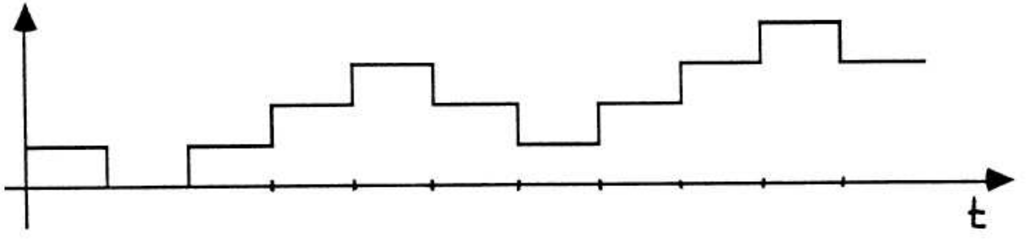
where φ is an RV that could be uniformly distributed in $(0,2\pi)$

L_1: Classic cases of RP's (2)

Further modeling is achieved by having the amplitude of the sinewave be an RV A(u) :

$$X(u,t) = A(u)\sin\left(2\pi f t + \phi(u)\right)$$

3. Random walk:



Manytimes we cannot describe the sample paths at all, neither in a mathematical nor a visual way → We can only give statistical information about the process

L_1: Statisitcal Description of RP's

The most general kind of stochastic information for a process is obtained by the joint probability distribution function of a number n of samples $X(u,t_1)$, $X(u,t_2)$,… for any n and any set $\{t_1,t_2,\ldots,t_n\}$

- Marginal PDF of one random variable:

$$F_X(x,t) \equiv Pr\{X(u,t) \leq x\}$$

- Joint PDF of two random variables obtained from the process by looking at time t_1 and t_2:

$$F(x_1,x_2;t_1,t_2) = Pr\{X(u,t_1) \leq x_1, X(u,t_2) \leq x_2\}$$

- In general, this is typically too demanding → Moments

L_1: Moments for real RP's

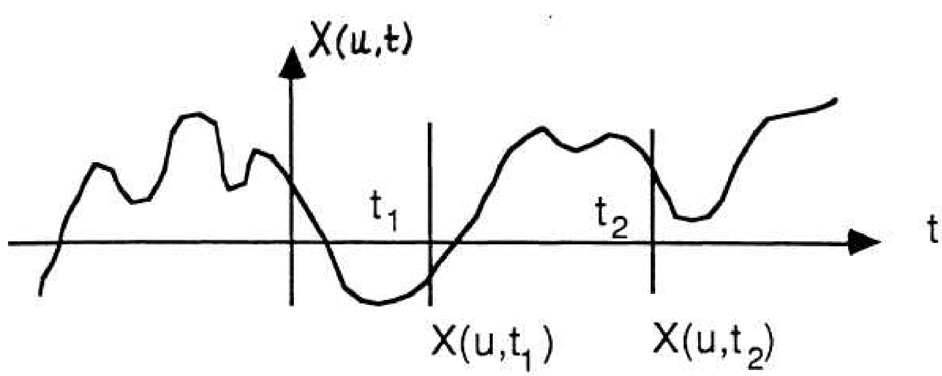
Mean value of *X(u,t)*: $m_X(t) \stackrel{\text{def}}{=} \mathcal{E}\{X(u,t)\}$

which is a deterministic function of *t*

Correlation function: $R_X(t_1,t_2) \stackrel{\text{def}}{=} \mathcal{E}\{X(u,t_1)X(u,t_2)\}$

which is again a deterministic function of two arguments t_1 and t_2.

The above definition for the autocorrelation holds for real *X(u,t)*.



L_1: Moments for complex RP's

If $X(u,t)$ is a complex process then we define:

$$R_X(t_1,t_2) \stackrel{\text{def}}{=} \mathcal{E}\{X(u,t_1)X^*(u,t_2)\}$$

where * means complex conjugate

Covariance function:

$$K_X(t_1,t_2) \stackrel{\text{def}}{=} \mathcal{E}\{(X(u,t_1) - m_X(t_1))(X(u,t_2) - m_X(t_2))^*\}$$

by expanding we conclude that

$$K_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)m_X(t_2)^*$$

Note: If $m_X(t) = 0$ then $K_x = R_x$. The above constitute second-order description of a random process, which very often is all we have or can calculate.

In general, knowing $m_X(t)$ and $R_X(t_1,t_2)$ says nothing about the underlying statistics which generated them. A notable exception is the *Gaussian* case.

L_1: Example of a mean value of a RP

1. $X(u,t) = A(u) \rightarrow m_X(t) = m_A = E\{A(u)\}$

<u>Note</u> : Suppose that $m_A = 0$, i.e. the *ensemble* average of *X(u,t)* is zero. Yet ,every time we do the experiment, (with probability 1 for continuous RV's) we see a constant number $\neq 0$! (for $-\infty < t < \infty$)

Here, the sample paths have little relation to the statistical averages of the process. Processes for which the sample path behavior relates to ensemble quantities are called *ergodic*.

L_1: Example of the first two moments of a RP (2)

2. Random sinusoid:

$$X(u,t) = \sin\left(t + \phi(u)\right)$$

$$\mathcal{E}\left\{X(u,t)\right\} = \int_0^{2\pi} \sin(t+\phi) f_\phi(\phi)\, d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(t+\phi)\, d\phi \qquad \forall t$$

$$R_X(t_1, t_2) = \mathcal{E}\left\{\sin(t_1+\phi)\sin(t_2+\phi)\right\}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(t_1+\phi)\sin(t_2+\phi)\, d\phi$$

$$= \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi} \left[\cos(t_1-t_2) - \cos(t_1+t_2+2\phi)\right] d\phi$$

$$= \frac{1}{2}\cos(t_1-t_2)$$

Notice that the autocorrelation depends only on the *difference* t_1 - t_2 and not t_1 or t_2 individually.

L_1: Correlation-function properties

Properties than any autocorrelation or covariance function must satisfy:

(a) Any well defined function $m_X(t)$ can be the mean function E{$X(u,t)$} of a process

(b) The correlation function $R_X(t_1,t_2)$ must be **Hermitian symmetric,** i.e.

$$R_X(t_1,t_2) = R_x^*(t_2,t_1)$$

(c) The correlation function must be a n*onnegative definite* function

L_1: Non-negative Definiteness + Schwartz

Definition: A complex function $R_X(t_1,t_2)$ is called *nonnegative definite* iff for any choice of *n* complex numbers $a_1, a_2, \ldots, a_n$ and every n-tuple $(t_1, t_2, \ldots, t_n)$> it is true that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a^*_j R_X(t_i, t_j) \geq 0$$

(d) The correlation function must satisfy the Schwartz inequality

$$|R_X(t_1,t_2)| \leq \sqrt{R_X(t_1,t_1)}\sqrt{R_X(t_2,t_2)}$$